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Michèle ASTAUD

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MM.	R. PALLU DE LA BARRIERE	Président
	N.N. BOGOLIUBOV	Examinateurs
	A. AVEZ	
	B. JOUVET	
	F. PERRIN	Invités
	A. VISCONTI	
	E. ARNOUS	

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BIBL.

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Renormalization « Groups » Extended to Masses; Application to the Structure of the Renormalization Constants Z_3 and m_0.

M. ASTAUD and B. JOUVET (*)

Laboratoire de Physique Atomique et Moléculaire, Collége de France - Paris

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Summary. — By a precise elucidation of the axioms from which renormalization « groups » result we construct the most general « groups » which include mass renormalization. We discuss in detail the particular case involving only photon field and electron mass renormalizations. An exact integral equation which can be put in a form generalizing that given by Gell-Mann and Low, allows us to show that a dependence of the bare charge α_0 on the observed one α is, contrary to an often quoted paradox, consistent with the renormalization group. We demonstrate that in the cases where the bare or the observed mass of the electron is null or infinite, which should correspond to determined values of α, the formal perturbation expansion in α_0 of the photon vacuum polarization tensor is no more divergent than a single power of a logarithm. We finally determine wherein lies the failure of the argument leading to the paradox stated above and discuss its implication.

Introduction.

Numerous works display the interesting consequences of the usual renormalization group (that we call $G(Z_3)$) of quantum electrodynamics [1], which, characterized by the simultaneous change $e^2 \to e^2/z_3$ and $D \to z_3 D$, leaves $e^2 D$

(*) Supported by the Commissariat à l'Énergie Atomique.

(1) M. GELL-MANN and F. LOW: *Phys. Rev.*, **95**, 1300 (1954); J. NILSSON: *Nuovo Cimento*, **39**, 84 (1965) and the references quoted therein.

invariant, D being the photon propagator. There were for a long time many indications that there exist other kinds of groups, or invariance structures, which have not been clearly expressed or studied and whose consequences, perhaps important, are completly unexplored:

1) Since the first field-theoretical studies of the π-nucleon interaction, it appeared that two different physical prescriptions could be used for defining the physical coupling constant G and correspondingly the renormalization point and the renormalization constant Z_1 of the vertex function: that of Kroll $(\Gamma_K(q^2=0)=1, G_K, Z_{1K})$ and that of Watson $(\Gamma_W(q^2=\mu^2)=1, G_W, Z_{1W})$; of course, to the extent that there exists a field theory (*i.e.* as long as bare fields and renormalization constants have a meaning), these two prescriptions, though they give different results at each stage of perturbation theory in G_K or G_W, are nevertheless equivalent for a complete solution, which, because it is a function of the bare constant G_0, can indifferently be expressed in terms of G_K or G_W related by

$$G_0 = G_K Z_{1K}/Z_2\sqrt{Z_3} = G_W Z_{1W}/Z_2\sqrt{Z_3}\,,$$

the renormalized vertices being such that $Z_{1W}\Gamma_W = Z_{1K}\Gamma_K$. By analogy with the well-known $G(Z_3)$ invariance structure we may call this one $G(Z_1^{-2})$.

2) More recently a cautious field-theoretical formulation of the Zachariasen model showed that a multiplicity of renormalization conditions led to the same physical amplitude ([2]), the invariance resulting there on the arbitrariness to decide what parts of the amplitude belong respectively to the vertex function or to the particle propagator.

This very particular example puts forward the notion, in fact obvious, that an arbitrary modification of the propagator's norm can be cancelled by a corresponding change of the vertex norm, providing the observed physical quantities can be expressed in terms of certain products of these functions (actually the product $\Gamma^2 D$); we call $G(Z_3 Z_1^{-2})$ such an invariance structure.

3) Lastly, as a result of the various investigations made to obtain a method of calculation of the function Z_3, it appeared that, although it is usual to express it in terms of α (or $\alpha_0 = \alpha/Z_3$) and of $(k^2/m^2)_{k^2\to\infty}$, where m is the electron physical mass, the value of the electron bare mass m_0 had a direct effect on that problem. JOHNSON *et al.* showed ([3]), on one side, that, if m_0 were null and Z_3 finite, the function Z_3 would at each order in α_0 be no more divergent than $\log k^2/m^2$. On the other side by studying the structure of the photon bare-

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([2]) J. C. HOUARD and J. C. LE GUILLOU: *Nuovo Cimento*, **44** A, 484 (1966).
([3]) K. JOHNSON, R. WILLEY and M. BAKER: *Phys. Rev.*, **163**, 1699 (1967).

mass expression, one of us obtained, under the condition that m_0 be finite, a complete expression of $Z_3(k^2/m_0^2, \alpha)$ which agrees with the known lowest orders of perturbation theory ([4]). However the photon propagator one deduced this way shows unitarity catastrophes at energies proportional to m_0, which indicates the importance of this constant which might have to be infinite on physical grounds.

Furthermore, the problem of studying precisely the role of the physical and bare masses in the renormalization group is in itself very important because one should know the degree of validity of the approximations which are always done in all the papers on that field, and also because it may be that the physically interesting theories are those for which the field renormalization constants are null ([5]), in which case the bare masses relay these constants in the asymptotic expression of the propagator. ($D(s)|_{s\to\infty} \to 1/\delta\nu^2$ instead of $D(s)|_{s\to\infty} \to (1/Z_3)s$.)

In a preceding paper ([6]) we reformulated and extended the theory of renormalization « groups », starting from various fundamental invariants; this method allows one to treat the most general group, $G(Z_3 Z_2 Z_2' Z_1^{-1} Z_1'^{-1})$, which connects change of renormalization of the charge and of the various key functions (propagators and vertices), and particularly the groups $G(Z_1^{-2})$ and $G(Z_3 Z_1^{-2})$ involved in the examples cited above. In that paper we treated in detail the « group » $G(Z_3 Z_2^2)$ and applied it for computing the asymptotic expression of certain classes of graphs.

The main object of the present paper is to extend once more the already extended groups of the previous paper, in such a way as to obtain the « group » properties associated with mass renormalization.

In Sect. 1 of this paper, a precise elucidation of the axioms from which the group derives will lead us to formulate a natural extension of the groups indicated, deduced in a restricted way in the previous paper, reserving for further publications the detailed consequences of these groups.

This new formulation will allow us in Sect. 2 of this paper to obtain in general various exact forms of integral equations for the photon propagator and the electron mass operator, and to study their simplifications when the

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([4]) *a*) B. JOUVET: *Nuovo Cimento*, **25**, 135 (1962); *b*) M. ASTAUD and B. JOUVET: *Compt. Rend.*, **263** B, 107 (1966).

([5]) See, for instance, J. C. HOUARD and B. JOUVET: *Nuovo Cimento*, **18**, 466 (1960).

([6]) M. ASTAUD and B. JOUVET: *Nuovo Cimento*, **53** A, 841 (1968). For other tentatives to extend the usual renormalization group, see K. E. ERIKSSON: *Nuovo Cimento*, **30**, 1423 (1963), who introduced through the use of a cut-off, a mass group which is different from that we present here. A. VISCONTI, who in a recent note (May, 1968, to be published) looks at the possible abstract mathematical group structures implied by the success of counter-terms + rescaling method for eliminating the perturbation theory divergences.

photon mass is null and the observed or the bare electron mass is null or infinite. One of the integral equations for the photon propagator takes a very useful form which, generalizing that given by GELL-MANN and LOW for its asymptotic domain, allows us, in Sect. 3, to discuss its global properties and to establish that, contrary to what has been often claimed (7), the existence of a relation between the values of the observed charge α and the bare charge α_0 is consistent with an exact treatment of the renormalization group. After verifying in Sect. 4 the exactness of the perturbation expansion of our integral equation for Z_3 and m_0 we show, in agreement with the proof given by JOHNSON *et al.* (3), that when the bare electron mass is null the perturbation expansion of $(Z_3-1)/\alpha$ in terms of α_0, though it is mathematically unjustified according to our result of Sect. 3, does not produce terms more divergent than a single power of a logarithm. We finally show in Sect. 5 how the classical paradoxical conclusion that α_0 is independent of α can be overcome and where its demonstration fails, our conclusion being that the many attempts to compute the coefficient of the logarithmic divergence are irrelevant in the problem of determining the exact asymptotic photon vacuum polarization tensor.

1. – General formulation of the « groups » G.

1·1. *Definitions.*

a) Let in the Maxwell-Dirac quantum electrodynamics be $D(k^2, e^2)$ the $\delta_{\mu\nu}$-part of the renormalized photon propagator, and μ and m the observed photon and electron masses. For dimensional reasons we can write

$$D(k, e^2) = \frac{1}{k^2} g\left(\frac{k^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right). \tag{1.1}$$

By definition the renormalized propagator has its pole at $k^2 = \mu^2$ with residue $=1$, then

$$\lim_{k^2 \to \mu^2} \left[g\left(\frac{k^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right)\right]^{-1} = 1 - \frac{\mu^2}{k^2} + o\big((k^2-\mu^2)^2\big); \tag{1.2}$$

$g(k^2/m^2, \mu^2/m^2, e^2)$ considered as a function of k^2 is real for $k^2 < 0$ and, if there is no violation of unitarity, it and its derivative are injective.

Let $S(p, e^2)$ be the renormalized electron propagator, which we may write as

$$S(p, e^2) = \left[(\gamma\cdot p) d_2\left(\frac{p^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) + m b\left(\frac{p^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right)\right]^{-1} \tag{1.3}$$

(7) J. D. BJORKEN and S. D. DRELL: *Relativistic Quantum Fields* (New York, 1965).

and which is such that

$$d_2\left(1, \frac{\mu^2}{m^2}, e^2\right) = -b\left(1, \frac{\mu^2}{m^2}, e^2\right) = 1 . \tag{1.4}$$

The functions d_2 and b are real for $p^2 < m^2$. Let us also call $d_1(k^2/m^2, p^2/m^2, (p-k)^2/m^2, \mu^2/m^2, e^2)$ the coefficient of γ_ϱ in the renormalized vertex function $\Gamma_\varrho(k, p, p-k)$. We will denote

$$d_1\left(\frac{k_{\mathrm{I}}^2}{m^2}, \frac{k_{\mathrm{II}}^2}{m^2}, \frac{k_{\mathrm{III}}^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) \equiv d_1\left(\frac{k^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right),$$

the variable k^2/m^2 when it occurs in d_1 being understood as a tri-vector. The renormalization condition is by definition

$$d_1\left(\frac{\mu^2}{m^2}, 1, 1, \frac{\mu^2}{m^2}, e^2\right) = 1 . \tag{1.5}$$

Let us now define the functions d_3, d_4, d_5 by

$$d_3\left(\frac{k^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) \equiv \frac{\partial}{\partial k^2} k^2 \left[g\left(\frac{k^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right)\right]^{-1}, \tag{1.6}$$

which is such that

$$d_3\left(\frac{\mu^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) = 1;$$

$$d_4\left(\frac{k^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) \equiv \frac{k^4}{\mu^2} \frac{\partial}{\partial k^2} \left[g\left(\frac{k^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right)\right]^{-1} \times \left[d_3\left(\frac{k^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right)\right]^{-1}, \tag{1.7}$$

which is such that

$$d_4\left(\frac{\mu^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) = 1;$$

$$d_5\left(\frac{k^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) \equiv \left[\frac{b(k^2/m^2, \mu^2/m^2, e^2)}{d_2(k^2/m^2, \mu^2/m^2, e^2)}\right]^2, \tag{1.8}$$

which is such that

$$d_5\left(1, \frac{\mu^2}{m^2}, e^2\right) = 1 .$$

Let us remark here that the function d_3 we introduced is different from the function g^{-1} with which one usually deals in the Z_3 renormalization group theory. Rather, the couple of functions d_3 and $\mu^2 d_4$ are respectively the slope of the inverse photon propagator at k^2, and the abscissa of the intersection of the tangent to this curve at this point with the k^2-axis. For this reason, such functions could equally be used when dealing with nonelementary particle propagators, such as Fermi coupling or $\lambda\cdot\varphi^4$ propagators, whether they have a composite pole or not.

Note that we have the relation

$$d_3\left(\frac{k^2}{m^2},\frac{\mu^2}{m^2},e^2\right)\times\left[1-\frac{\mu^2}{k^2}d_4\left(\frac{k^2}{m^2},\frac{\mu^2}{m^2},e^2\right)\right]=\left[g\left(\frac{k^2}{m^2},\frac{\mu^2}{m^2},e^2\right)\right]^{-1}, \tag{1.9}$$

which can also be written

$$d_4\left(\frac{k^2}{m^2},\frac{\mu^2}{m^2},e^2\right)d_3\left(\frac{k^2}{m^2},\frac{\mu^2}{m^2},e^2\right)= \tag{1.10}$$

$$=\frac{k^4}{\mu^2}\frac{\partial}{\partial k^2}\left[d_3\left(\frac{k^2}{m^2},\frac{\mu^2}{m^2},e^2\right)\left[1-\frac{\mu^2}{k^2}d_4\left(\frac{k^2}{m^2},\frac{\mu^2}{m^2},e^2\right)\right]\right].$$

We finally introduced five functions $d_i(k^2/m^2,\mu^2/m^2,e^2)$; k^2/m^2 is a scalar for $i=2,\ldots,5$ and a tri-vector for $i=1$.

b) The same Maxwell-Dirac quantum-electrodynamic theory can also be formulated in terms of unrenormalized quantities, m_0 and e_0, the bare mass and charge of the electron, μ_0 the photon bare mass, D_0 and S_0 the unrenormalized photon and electron propagators, $\Gamma_{0\varrho}$ the unrenormalized vertex.

As in the case of renormalized functions, we can define the unrenormalized functions

$$\begin{cases} k^2 D_0 = g_0\left(\dfrac{k^2}{m_0^2},\dfrac{\mu_0^2}{m_0^2},e_0^2\right), \\ S_0 \;\; = \left[(\gamma\cdot p)d_{02}\left(\dfrac{p^2}{m_0^2},\dfrac{\mu_0^2}{m_0^2},e_0^2\right)+m_0 b_0\left(\dfrac{p^2}{m_0^2},\dfrac{\mu_0^2}{m_0^2},e_0^2\right)\right]^{-1}, \\ d_{03} \;\; \equiv \dfrac{\partial}{\partial k^2}k^2 g_0^{-1}, \\ d_{04} \;\; \equiv d_{03}^{-1}\dfrac{k^4}{\mu_0^2}\dfrac{\partial}{\partial k^2}g_0^{-1}, \\ d_{05} \;\; \equiv \left(\dfrac{b_0}{d_{02}}\right)^2, \qquad d_{01}\equiv \text{coefficient of } \gamma_\varrho \text{ in } \Gamma_{0\varrho}. \end{cases} \tag{1.11}$$

Dyson's formulae relate the renormalized functions to the unrenormalized functions through the renormalization constants Z_1, Z_2, Z_3, and mass renormalization constants, which for convenience we take in the form

$$Z_4 = \frac{\mu_0^2}{\mu^2} \quad \text{and} \quad Z_5 = \frac{m_0^2}{m^2}; \tag{1.12}$$

$$\begin{cases} S_0(k, \mu_0, m_0, e_0^2) = Z_2 S(k, \mu, m, e^2) , \\ D_0(k, \mu_0, m_0, e_0^2) = Z_3 D(k, \mu, m, e^2) , \\ \Gamma_0(k, p, p-k, \mu_0, m_0 \ e_0) = Z_1 \Gamma(k, p, p-k, \mu, m, e) . \end{cases} \tag{1.13}$$

One easily verifies that

$$d_{0i}\left(\frac{k^2}{m_0^2}, \frac{\mu_0^2}{m_0^2}, e_0^2\right) = Z_i^{-1} d_i\left(\frac{k^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) \qquad (\forall i = 1, 2, 3, 4, 5) . \tag{1.14}$$

The constants Z_2, Z_3 are the residues of the u.r. propagators S_0 and D_0, and Z_1 the value of the u.r. vertex function when the particles are on shell.

Through the relation (1.14), Z_1, Z_2 Z_3 are defined as functions of the bare constants; also μ^2 and m^2 being the poles of the renormalized propagators allows one to express Z_4 and Z_5 in terms of the bare constants.

c) The Lehmann sum rules, resulting (when canonical relations, causality, covariance and unitarity are assumed) from the Lagrangian expressed in terms of the physical fields and the renormalization constants, allow one to relate the renormalization constants Z_2 and Z_3 and the bare masses m_0 and μ_0 to the asymptotic form of the renormalized propagators:

$$\begin{cases} \lim\limits_{k^2 \to -\infty} [k^2 D(k^2, e^2)]^{-1} = Z_3 \left[1 - \frac{\mu_0^2}{k^2} + o\left(\frac{1}{k^4}\right)\right] , \\ \lim\limits_{p^2 \to -\infty} [S(p, e^2)]^{-1} \quad = Z_2 \left[(\gamma \cdot p)\left(1 + o_1\left(\frac{1}{p_2}\right)\right) - m_0\left(1 + o_2\left(\frac{1}{p_2}\right)\right)\right] . \end{cases} \tag{1.15}$$

Thereby the renormalized constants can be expressed in terms of the physical quantities m, μ, e, through the relations

$$\lim_{k^2 \to -\infty} d_i\left(\frac{k^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) = Z_i , \qquad i = 2, 3, 4, 5 . \tag{1.16a}$$

In accordance with the definition of Z_1 given in (1.5) (which is equal to Z_2),

we can also express

$$\lim_{k^2\to-\infty} d_1\left(\frac{\mu^2}{m^2}, \frac{k^2}{m^2}, \frac{k^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) = Z_1 . \tag{1.16b}$$

If the renormalized vacuum polarization $\Pi(k^2, e^2)$ is defined by

$$D \equiv [k^2(1 - e^2\Pi)]^{-1} , \tag{1.17}$$

one has also

$$\lim_{k^2\to-\infty} [1 - e^2\Pi(k^2, e^2)] = Z_3 , \tag{1.18}$$

and therefore according to (1.16a) d_3 and $1 - e^2\Pi(k^2, e^2)$ are of the same order of magnitude when $k^2 \to -\infty$.

1·2. *Family of interpolating functions.* – Let us define the function $\hat{d}_i$ by

$$\hat{d}_i\left(\frac{k^2}{m^2}, \frac{\lambda^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) \equiv \frac{d_i(k^2/m^2, \mu^2/m^2, e^2)}{d_i(\lambda^2/m^2, \mu^2/m^2, e^2)} . \tag{1.19}$$

For i given, the functions $\hat{d}_i$ depend on one or three parameters; they form a « family » F_i, to which belong the functions d_i (that we call « renormalized ») and the functions d_{i0} (that we call « bare »); indeed one has

$$\left|\begin{array}{l} \hat{d}_i\left(\dfrac{k^2}{m^2}, \dfrac{\mu^2}{m^2}, \dfrac{\mu^2}{m^2}, e^2\right) \equiv d_i\left(\dfrac{k^2}{m^2}, \dfrac{\mu^2}{m^2}, e^2\right) , \\ \qquad\qquad \text{for } i = 3,4 \text{ from (1.6) and (1.7)}, \\ \hat{d}_i\left(\dfrac{k^2}{m^2}, 1, \dfrac{\mu^2}{m^2}, e^2\right) \equiv d_i\left(\dfrac{k^2}{m^2}, \dfrac{\mu^2}{m^2}, e^2\right) , \\ \qquad\qquad \text{for } i = 2,5 \text{ from (1.4) and (1.8)} , \\ \hat{d}_1\left(\dfrac{k^2}{m^2}, \dfrac{\mu^2}{m^2}, 1, 1, \dfrac{\mu^2}{m^2}, e^2\right) \equiv d_1\left(\dfrac{k^2}{m^2}, \dfrac{\mu^2}{m^2}, e^2\right) , \quad \text{for } i = 1 \text{ (*) from (1.5)}; \end{array}\right. \tag{1.20}$$

$$\left|\begin{array}{l} \hat{d}_i\left(\dfrac{k^2}{m^2}, \dfrac{-\infty}{m^2}, \dfrac{\mu^2}{m^2}, e^2\right) = Z_i^{-1} d_i\left(\dfrac{k^2}{m^2}, \dfrac{\mu^2}{m^2}, e^2\right) = d_{0i}\left(\dfrac{k^2}{m_0^2}, \dfrac{\mu_0^2}{m_0^2}, e_0^2\right) , \\ \qquad\qquad \text{for } i = 2, 3, 4, 5, \text{ from (1.16}a\text{)}, \\ \hat{d}_1\left(\dfrac{k^2}{m^2}, \dfrac{\mu^2}{m^2}, \dfrac{-\infty}{m^2}, \dfrac{-\infty}{m^2}, e^2\right) = Z_1^{-1} d_1\left(\dfrac{k^2}{m^2}, \dfrac{\mu^2}{m^2}, e^2\right) = d_{01}\left(\dfrac{k^2}{m_0^2}, \dfrac{\mu_0^2}{m_0^2}, e_0^2\right) , \\ \qquad\qquad \text{for } i = 1 \text{ from (1.16}b\text{)}. \end{array}\right. \tag{1.21}$$

(*) We recall that for $i = 1$

$$\hat{d}_1\left(\frac{k^2}{m^2}, \frac{\lambda^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) \equiv \hat{d}_1\left(\frac{k_{\mathrm{I}}^2}{m^2}, \frac{k_{\mathrm{II}}^2}{m^2}, \frac{k_{\mathrm{III}}^2}{m^2}, \frac{\lambda_{\mathrm{I}}^2}{m^2}, \frac{\lambda_{\mathrm{II}}^2}{m^2}, \frac{\lambda_{\mathrm{III}}^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) .$$

Two functions $\hat{d}_i(k^2/m^2, \lambda^2/m^2, \mu^2/m^2, e^2)$ and $\hat{d}_i(k^2/m^2, \lambda'^2/m^2, \mu^2/m^2, e^2)$ are related by the equation

$$\hat{d}_i\left(\frac{k^2}{m^2}, \frac{\lambda^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) = \hat{d}_i\left(\frac{k^2}{m^2}, \frac{\lambda'^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) \times \hat{d}_i\left(\frac{\lambda'^2}{m^2}, \frac{\lambda^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right), \tag{1.22}$$

which is an obvious consequence of the definition (1.19). Conversely, defining the d_i by (1.20), the solutions of (1.22) are (1.19).

We note that (1.22) gives $\hat{d}_i(\lambda'^2/m^2, \lambda'^2/m^2, \mu^2/m^2, e^2) = 1$, therefore the property of the functions $\hat{d}_i$ to be normalized at $k^2/m^2 = \lambda^2/m^2$ is contained in (1.22).

If we consider the functions $X_{i_\lambda} \equiv d_i(k^2/m^2, \lambda^2/m^2, \mu^2/m^2, e^2)$ of the variable λ^2/m^2 (the other variables being fixed) one has the transformation law

$$X_{i_\lambda} = X_{i_{\lambda'}} T_{i_{\lambda'\lambda}} \qquad \text{with} \qquad T_{i_{\lambda'\lambda}} \equiv \hat{d}_i\left(\frac{\lambda'^2}{m^2}, \frac{\lambda^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right). \tag{1.23}$$

That transformations satisfies the composition law

$$T_{i_{\lambda'\lambda}} = T_{i_{\lambda'\lambda''}} \cdot T_{i_{\lambda''\lambda}}, \tag{1.24}$$

which is commutative but does not permit us to compose

$$T_{i_{\lambda''\lambda'}} \cdot T_{i_{\lambda'''\lambda}} \qquad \text{if} \qquad \lambda'' \neq \lambda \text{ and } \lambda' \neq \lambda'''.$$

The five families F_i of functions $\hat{d}_i$ are called a « group » G.

1˙3. *Change of variables.* – Let us make, in the set of functions $\hat{d}_i(k^2/m^2, \lambda^2/m^2, \mu^2/m^2, e^2)$, k^2 and λ^2 being conserved, the change of variables

$$C \equiv C^{\mu_{m(\mu m m)}^{m(\mu m m)\mu m}}_{\theta^{\nu\,(\tau)}_{\omega\,(\varrho)}\eta\zeta} \qquad (e^2, \mu^2, m^2) \to (e^2_{\theta^{\nu\tau}_{\omega\varrho}}\ \mu^2_\eta\ m^2_\zeta), \tag{1.25}$$

the new interpolating variables being expressed in terms of the old ones by the formulae

$$\begin{cases} e^2_{\theta^{\nu\tau}_{\omega\varrho}} = e^2 \dfrac{d_1(\tau_{\mathrm{I}}^2/m^2, \tau_{\mathrm{II}}^2/m^2, \tau_{\mathrm{III}}^2/m^2, \mu^2/m^2, e^2)\ d_1(\varrho_{\mathrm{I}}^2/m^2, \varrho_{\mathrm{II}}^2/m^2, \varrho_{\mathrm{III}}^2/m^2, \mu^2/m^2, e^2)}{d_3(\theta^2/m^2, \mu^2/m^2, e^2) \cdot d_2(\nu^2/m^2, \mu^2/m^2, e^2) \cdot d_2(\omega^2/m^2, \mu^2/m^2, e^2)}, \\ \mu^2_\eta = \mu^2 d_4\left(\dfrac{\eta^2}{m^2}, \dfrac{\mu^2}{m^2}, e^2\right), \\ m^2_\zeta = m^2 d_5\left(\dfrac{\zeta^2}{m^2}, \dfrac{\mu^2}{m^2}, e^2\right) (*). \end{cases} \tag{1.26}$$

(*) d_5 is not necessarily injective, so it may happen that $m_{\zeta_1} = m_{\zeta_2}$ for $\zeta_1 \neq \zeta_2$.

Let us note that this particular choice, which makes a link between the five classes of functions $\hat{d}_i$, $i=1,\ldots,5$, of the family, is suitably chosen in such a way that the interpolating « masses » of the photon and of the electron μ_η^2 and m_ζ^2, and the interpolating « charge » $e^2_{\theta^{\nu\tau}_{\omega\varrho}}$ are equal, for certain values of the parameters of C, to the « renormalized » or to the « bare » masses and charges; indeed one verifies that

$$(1.27)\qquad \begin{cases} e^2_{\mu^{m(\mu mm)}_{m(\mu mm)}} = e^2\,, \qquad e^2_{\infty^{\infty(\mu\infty\infty)}_{\infty(\mu\infty\infty)}} = e_0^2\,, \\ \mu^2_\mu = \mu^2\,, \qquad \mu^2_\infty = \mu_0^2\ (^*)\,, \qquad m^2_m = m^2\,, \qquad m^2_\infty = m_0^2\,. \end{cases}$$

1·4. *Functional equations.* – In order that C be a change of variables, it is necessary that one be able, by inverting the eqs. (1.26), to express the old variables in terms of the new ones.

e^2, μ^2/μ_η^2, m^2/m_ζ^2 are then functions of the variables

$$\mathscr{E}\qquad \left(\frac{\nu^2}{\theta^2}, \frac{\tau^2}{\theta^2}, \frac{\omega^2}{\theta^2}, \frac{\varrho^2}{\theta^2}, \frac{\eta^2}{\theta^2}, \frac{\zeta^2}{\theta^2}, \frac{\mu_\eta^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right),$$

among which we distinguish the set $\langle E\rangle = (\nu^2/\theta^2, \tau^2/\theta^2, \omega^2/\theta^2, \varrho^2/\theta^2, \eta^2/\theta^2, \zeta^2/\theta^2)$ of parameters involved in the change of variables C. Then

$$(1.28)\qquad z_i\left(\frac{k^2}{\theta^2}, \frac{\mu_\eta^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, \frac{\lambda^2}{\theta^2}, \langle E\rangle, e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right) = \hat{d}_i\left(\frac{k^2}{m^2(\mathscr{E})}, \frac{\lambda^2}{m^2(\mathscr{E})}, \frac{\mu^2(\mathscr{E})}{m^2(\mathscr{E})}, e^2(\mathscr{E})\right)$$

is a functional form of a family of functions depending on the variables k^2, μ_η^2, m_ζ^2, $e^2_{\theta^{\nu\tau}_{\omega\varrho}}$, on a parameter (or tri-parameter) λ^2 of normalization, and on the set $\langle E\rangle$ of parameters which characterizes the change of variables by means of which the family is constructed. (One could equally well have written $z_{i\lambda;\theta^{\nu\tau}_{\omega\varrho}\eta\zeta}(k^2/m_\zeta^2, \mu_\eta^2/m_\zeta^2, e^2_{\theta^{\nu\tau}_{\omega\varrho}})$, the indices indicating that we have different functions for different values of the indices.)

Another change of variables $C' = C^{\mu^{m(\mu mm)}_{m(\mu mm)}\mu m}_{\theta'^{\nu'\tau'}_{\omega'\varrho'}\eta'\zeta'}$ would give of course

$$(1.29)\qquad z_i\left(\frac{k^2}{\theta'^2}, \frac{\mu_{\eta'}^2}{\theta'^2}, \frac{m_{\zeta'}^2}{\theta'^2}, \frac{\lambda^2}{\theta'^2}, \langle E'\rangle, e^2_{\theta'^{\nu'\tau'}_{\omega'\varrho'}}\right) = \hat{d}_i\left(\frac{k^2}{m^2(\mathscr{E}')}, \frac{\lambda^2}{m^2(\mathscr{E}')}, \frac{\mu^2(\mathscr{E}')}{m^2(\mathscr{E}')}, e^2(\mathscr{E}')\right).$$

But since

$$(1.30)\qquad \hat{d}_i\left(\frac{k^2}{m^2}, \frac{\lambda^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) = \hat{d}_i\left(\frac{k^2}{m^2(\mathscr{E})}, \frac{\lambda^2}{m^2(\mathscr{E})}, \frac{\mu^2(\mathscr{E})}{m^2(\mathscr{E})}, e^2(\mathscr{E})\right) = \hat{d}_i\left(\frac{k^2}{m^2(\mathscr{E}')}, \frac{\lambda^2}{m^2(\mathscr{E}')}, \frac{\mu^2(\mathscr{E}')}{m^2(\mathscr{E}')}, e^2(\mathscr{E}')\right),$$

(*) Of course μ^2_∞ means μ^2 for $\eta^2 = -\infty$ $(\eta = i\infty)$.

one has

$$(1.31)\qquad z_i\left(\frac{k^2}{\theta^2},\frac{\mu_\eta^2}{\theta^2},\frac{m_\zeta^2}{\theta^2},\frac{\lambda^2}{\theta^2},\langle E\rangle,e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right)=z_i\left(\frac{k^2}{\theta'^2},\frac{\mu_{\eta'}^2}{\theta'^2},\frac{m_{\zeta'}^2}{\theta'^2},\frac{\lambda^2}{\theta'^2},\langle E'\rangle,e^2_{\theta'^{\nu'\tau'}_{\omega'\varrho'}}\right),$$

the same quantity being expressed in terms of different variables.

So far we made the changes of variables $C^{m(\mu mm)\mu m}_{m(\mu mm)\mu m\,\theta^{\nu\tau}_{\omega\varrho}\eta\zeta}$ or $C^{m(\mu mm)\mu m}_{m(\mu mm)\mu m\,\theta'^{\nu'\tau'}_{\omega'\varrho'}\eta'\zeta'}$ but we can also consider the charge of variables $C^{\theta^{\nu\tau}_{\omega\varrho}\eta\zeta}_{\theta'^{\nu'\tau'}_{\omega'\varrho'}\eta'\zeta}$ which using (1.26), (1.19) and (1.28) is defined by

$$(1.32)\qquad\begin{cases} e^2_{\theta'^{\nu'\tau'}_{\omega'\varrho'}}=e^2_{\theta^{\nu\tau}_{\omega\varrho}}\dfrac{z_1\left(\frac{\tau'^2}{\theta^2},\frac{\mu_\eta^2}{\theta^2},\frac{m_\zeta^2}{\theta^2},\frac{\tau^2}{\theta^2},\langle E\rangle,e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right)}{z_3\left(\frac{\theta'^2}{\theta^2},\frac{\mu_\eta^2}{\theta^2},\frac{m_\zeta^2}{\theta^2},1,\langle E\rangle,e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right)}\cdot\\ \qquad\qquad\cdot\dfrac{z_4\left(\frac{\varrho'^2}{\theta^2},\frac{\mu_\eta^2}{\theta^2},\frac{m_\zeta^2}{\theta^2},\frac{\varrho^2}{\theta^2},\langle E\rangle,e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right)}{z_2\left(\frac{\nu'^2}{\theta^2},\frac{\mu_\eta^2}{\theta^2},\frac{m_\zeta^2}{\theta^2},\frac{\nu^2}{\theta^2},\langle E\rangle,e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right)\cdot z_2\left(\frac{\omega'^2}{\theta^2},\frac{\mu_\eta^2}{\theta^2},\frac{m_\zeta^2}{\theta^2},\frac{\omega^2}{\theta^2},\langle E\rangle,e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right)},\\ \mu_{\eta'}^2=\mu_\eta^2 z_4\left(\frac{\eta'^2}{\theta^2},\frac{\mu_\eta^2}{\theta^2},\frac{m_\zeta^2}{\theta^2},\frac{\eta^2}{\theta^2},\langle E\rangle,e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right),\\ m_{\zeta'}^2=m_\zeta^2 z_5\left(\frac{\zeta'^2}{\theta^2},\frac{\mu_\eta^2}{\theta^2},\frac{m_\zeta^2}{\theta^2},\frac{\zeta^2}{\theta^2},\langle E\rangle,e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right).\end{cases}$$

Expressing $e^2_{\theta'^{\nu'\tau'}_{\omega'\varrho'}}$, $\mu^2_{\eta'}$, $m^2_{\zeta'}$ by (1.32) in (1.31) one gets thereby an *identity in the variables* k^2, λ^2, μ_η^2, m_ζ^2, $e^2_{\theta^{\nu\tau}_{\omega\varrho}}$, θ^2, ν^2, τ^2, ω^2, ϱ^2, η^2, ζ^2, θ'^2, ν'^2, τ'^2, ω'^2, ϱ'^2, η'^2, ζ'^2.

The identity in the five first variables is the consequence of the fact that we cancelled a change of variables by an inverse one; the identity in the other variables is obvious because, these parameters being independent, (1.31) cannot be a relation among them.

This identity is a functional equation between the different functions z_i, in the same way that $\exp[\log x]=x$ is an identity in x but a functional relation between the exponential and the logarithmic functions.

Other functional equations can also be obtained by changing the variables in (1.22) (which is already a functional equation, identity in k^2, λ^2, λ'^2, m^2, μ^2, e^2)

$$(1.33)\qquad \hat{d}_i\left(\frac{k^2}{m^2(\mathscr{E})},\frac{\lambda^2}{m^2(\mathscr{E})},\frac{\mu^2(\mathscr{E})}{m^2(\mathscr{E})},e^2(\mathscr{E})\right)=$$

$$=\hat{d}_i\left(\frac{k^2}{m^2(\mathscr{E}')},\frac{\lambda'^2}{m^2(\mathscr{E}')},\frac{\mu^2(\mathscr{E}')}{m^2(\mathscr{E}')},e^2(\mathscr{E}')\right)\cdot\hat{d}_i\left(\frac{\lambda'^2}{m^2(\mathscr{E}'')},\frac{\lambda^2}{m^2(\mathscr{E}'')},\frac{\mu^2(\mathscr{E}'')}{m^2(\mathscr{E}'')},e^2(\mathscr{E}'')\right).$$

Taking $\mathscr{E}''=\mathscr{E}$ in (1.33) we get

$$(1.34)\qquad \widehat{d}_i\left(\frac{k^2}{m^2(\mathscr{E})},\frac{\lambda^2}{m^2(\mathscr{E})},\frac{\mu^2(\mathscr{E})}{m^2(\mathscr{E})},e^2(\mathscr{E})\right)= -\widehat{d}_i\left(\frac{k^2}{m^2(\mathscr{E}')},\frac{\lambda'^2}{m^2(\mathscr{E}')},\frac{\mu^2(\mathscr{E}')}{m^2(\mathscr{E}')},e^2(\mathscr{E}')\right)\cdot\widehat{d}_i\left(\frac{\lambda'^2}{m^2(\mathscr{E})},\frac{\lambda^2}{m^2(\mathscr{E})},\frac{\mu^2(\mathscr{E})}{m^2(\mathscr{E})},e^2(\mathscr{E})\right);$$

one has for $k^2=\lambda'^2$ in (1.34)

$$(1.35)\qquad \widehat{d}_i\left(\frac{\lambda'^2}{m^2(\mathscr{E}')},\frac{\lambda'^2}{m^2(\mathscr{E}')},\frac{\mu^2(\mathscr{E}')}{m^2(\mathscr{E}')},e^2(\mathscr{E}')\right)=1\,.$$

Formulae (1.34) with $\lambda'^2=\lambda^2$ and (1.35) give (1.30); from (1.30) we get

$$\widehat{d}_i\left(\frac{\lambda'^2}{m^2(\mathscr{E})},\frac{\lambda^2}{m^2(\mathscr{E})},\frac{\mu^2(\mathscr{E})}{m^2(\mathscr{E})},e^2(\mathscr{E})\right)=\widehat{d}_i\left(\frac{\lambda'^2}{m^2(\mathscr{E}'')},\frac{\lambda^2}{m^2(\mathscr{E}'')},\frac{\mu^2(\mathscr{E}'')}{m^2(\mathscr{E}'')},e^2(\mathscr{E}'')\right),$$

which inserted in (1.34) gives (1.33) therefore (1.33) $\Leftrightarrow$ (1.34).

Also (1.30) and (1.35) follow from (1.34).

Writing (1.34) in terms of the z_i

$$(1.36)\qquad z_i\left(\frac{k^2}{\theta^2},\frac{\mu_\eta^2}{\theta^2},\frac{m_\xi^2}{\theta^2},\frac{\lambda^2}{\theta^2},\langle E\rangle,e^2_{\theta\omega\varrho}{}^{\nu\tau}\right)= z_i\left(\frac{k^2}{\theta'^2},\frac{\mu_{\eta'}^2}{\theta'^2},\frac{m_{\xi'}^2}{\theta'^2},\frac{\lambda'^2}{\theta'^2},\langle E'\rangle,e^2_{\theta'\omega'\varrho'}{}^{\nu'\tau'}\right)\cdot z_i\left(\frac{\lambda'^2}{\theta^2},\frac{\mu_\eta^2}{\theta^2},\frac{m_\xi^2}{\theta^2},\frac{\lambda^2}{\theta^2},\langle E\rangle,e^2_{\theta\omega\varrho}{}^{\nu\tau}\right).$$

When $e^2_{\theta'\omega'\varrho'}{}^{\nu'\tau'}$, $\mu_{\eta'}^2$, $m_{\xi'}^2$ are replaced in (1.36) by their expressions (1.32) one gets functional equations (identity in the variables) which are our fundamental equations $i=1,2,3,4,5$:

$$(1.37)\qquad z_i\left(\frac{k^2}{\theta^2},\frac{\mu_\eta^2}{\theta^2},\frac{m_\xi^2}{\theta^2},\frac{\lambda^2}{\theta^2},\langle E\rangle,e^2_{\theta\omega\varrho}{}^{\nu\tau}\right)= z_i\left(\frac{\lambda'^2}{\theta^2},\frac{\mu_\eta^2}{\theta^2},\frac{m_\xi^2}{\theta^2},\frac{\lambda^2}{\theta^2},\langle E\rangle,e^2_{\theta\omega\varrho}{}^{\nu\tau}\right)\cdot$$

$$\cdot z_i\left(\frac{k^2}{\theta'^2},\frac{\mu_\eta^2}{\theta'^2}z_4\left(\frac{\eta'^2}{\theta^2},\frac{\mu_\eta^2}{\theta^2},\frac{m_\xi^2}{\theta^2},\frac{\eta^2}{\theta^2},\langle E\rangle,e^2_{\theta\omega\varrho}{}^{\nu\tau}\right),\frac{m_\xi^2}{\theta'^2}z_5\left(\frac{\zeta'^2}{\theta^2},\frac{\mu_\eta^2}{\theta^2},\frac{m_\xi^2}{\theta^2},\frac{\zeta^2}{\theta^2},\langle E\rangle,e^2_{\theta\omega\varrho}{}^{\nu\tau}\right),\right.$$

$$\left.\frac{\lambda'^2}{\theta'^2},\langle E'\rangle,e^2_{\theta\omega\varrho}{}^{\nu\tau}\,\frac{z_1\left(\frac{\tau'^2}{\theta^2},\frac{\mu_\eta^2}{\theta^2},\frac{m_\xi^2}{\theta^2},\frac{\tau^2}{\theta^2},\langle E\rangle,e^2_{\theta\omega\varrho}{}^{\nu\tau}\right)}{z_3\left(\frac{\theta'^2}{\theta^2},\frac{\mu_\eta^2}{\theta^2},\frac{m_\xi^2}{\theta^2},1,\langle E\rangle,e^2_{\theta'\omega\varrho}{}^{\nu\tau}\right)}\cdot\right.$$

$$\left.\cdot\frac{z_1\left(\frac{\varrho'^2}{\theta^2},\frac{\mu_\eta^2}{\theta^2},\frac{m_\xi^2}{\theta^2},\frac{\varrho^2}{\theta^2},\langle E\rangle,e^2_{\theta\omega\varrho}{}^{\nu\tau}\right)}{z_2\left(\frac{\nu'^2}{\theta^2},\frac{\mu_\eta^2}{\theta^2},\frac{m_\xi^2}{\theta^2},\frac{\nu^2}{\theta^2},\langle E\rangle,e^2_{\theta\omega\varrho}{}^{\nu\tau}\right)\cdot z_2\left(\frac{\omega'^2}{\theta^2},\frac{\mu_\eta^2}{\theta^2},\frac{m_\xi^2}{\theta^2},\frac{\omega^2}{\theta^2},\langle E\rangle,e^2_{\theta\omega\varrho}{}^{\nu\tau}\right)}\right).$$

One notes, however, that, contrary to (1.36) when $k^2 = \lambda'^2$, (1.37) does not give

$$z_i\left(\frac{\lambda^2}{\theta^2}, \frac{\mu_\eta^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, \frac{\lambda^2}{\theta^2}, \langle E\rangle, e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right) = 1 . \tag{1.38}$$

Therefore, when using in what follows our fundamental set of functional equations (1.37) one should also impose (1.38). Taking $\langle E'\rangle = \langle E\rangle$ and $\theta' = \theta$ in (1.37) one gets

$$z_i\left(\frac{k^2}{\theta^2}, \frac{\mu_\eta^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, \frac{\lambda^2}{\theta^2}, \langle E\rangle, e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right) =$$

$$= z_i\left(\frac{\lambda'^2}{\theta^2}, \frac{\mu_\eta^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, \frac{\lambda^2}{\theta^2}, \langle E\rangle, e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right) \cdot z_i\left(\frac{k^2}{\theta^2}, \frac{\mu_\eta^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, \frac{\lambda'^2}{\theta^2}, \langle E\rangle, e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right),$$

the general solution of which is

$$z_i\left(\frac{k^2}{\theta^2}, \frac{\mu_\eta^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, \frac{\lambda^2}{\theta^2}, \langle E\rangle, e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right) = \frac{f_i(k^2/\theta^2, \mu_\eta^2/\theta^2, m_\zeta^2/\theta^2, \langle E\rangle, e^2_{\theta^{\nu\tau}_{\omega\varrho}})}{f_i(\lambda^2/\theta^2, \mu_\eta^2/\theta^2, m_\zeta^2/\theta^2, \langle E\rangle, e^2_{\theta^{\nu\tau}_{\omega\varrho}})}, \tag{1.39}$$

where f_i is an arbitrary function, which contains fewer variables than z_i, making easier the finding of the solutions of (1.37).

Remarks.

I) One has

$$\left\{\begin{aligned}
&e^2 z_1 z_1' z_2^{-1} z_2'^{-1} z_3^{-1} \equiv e^2_{\theta^{\nu\tau}_{\omega\varrho}} \frac{z_1\left(\frac{k^2}{\theta^2}, \frac{\mu_\eta^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, \frac{\tau^2}{\theta^2}, \langle E\rangle, e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right)}{z_3\left(\frac{p^2}{\theta^2}, \frac{\mu_\eta^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, 1, \langle E\rangle, e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right)} \cdot \\
&\cdot \frac{z_1\left(\frac{k'^2}{\theta^2}, \frac{\mu_\eta^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, \frac{\varrho^2}{\theta^2}, \langle E\rangle, e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right)}{z_2\left(\frac{q^2}{\theta^2}, \frac{\mu_\eta^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, \frac{\nu^2}{\theta^2}, \langle E\rangle, e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right) \cdot z_2\left(\frac{q'^2}{\theta^2}, \frac{\mu_\eta^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, \frac{\omega^2}{\theta^2}, \langle E\rangle, e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right)} = \\
&\qquad - e^2 \frac{d_1\left(\frac{k^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) \cdot d_1\left(\frac{k'^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right)}{d_3\left(\frac{p^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) \cdot d_2\left(\frac{q^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) \cdot d_2\left(\frac{q'^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right)}, \\
&\mu^2 z_4 - \mu_\eta^2 z_4\left(\frac{k^2}{\theta^2}, \frac{\mu_\eta^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, \frac{\eta^2}{\theta^2}, \langle E\rangle, e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right) = \mu^2 d_4\left(\frac{k^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right), \\
&m^2 z_5 = m_\zeta^2 z_5\left(\frac{k^2}{\theta^2}, \frac{\mu_\eta^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, \frac{\zeta^2}{\theta^2}, \langle E\rangle, e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right) = m^2 d_5\left(\frac{k^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right),
\end{aligned}\right. \tag{1.40}$$

and therefore the quantities $e^2 z_1 z_1'^{-1} z_2^{-1} z_2'^{-1} z_3^{-1}$, $\mu^2 z_4$, $m^2 z_5$ are independent of the parameters $\langle E \rangle$: they are the invariants of the « group » $G(Z_3 Z_1^{-1} Z_1'^{-1} Z_2 Z_2', \mu, m)$ defined in the next Section.

II) Dividing (1.10) by $d_4(\eta^2/m^2, \mu^2/m^2, e^2) \cdot d_3(\eta^2/m^2, \mu^2/m^2, e^2)$ and using the functions defined by (1.19) one gets the relation

$$\hat{d}_4\left(\frac{k^2}{m^2}, \frac{\eta^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) \hat{d}_3\left(\frac{k^2}{m^2}, \frac{\eta^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right) =$$

$$= -k^4 \frac{\partial}{\partial k^2}\left[d_3\left(\frac{k^2}{m^2}, \frac{\eta^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right)\left\{\frac{1}{\mu^2 d_4(\eta^2/m^2, \mu^2/m^2, e^2)} - \frac{\hat{d}_4(k^2/m^2, \eta^2/m^2, \mu^2/m^2, e^2)}{k^2}\right\}\right],$$

which, expressed in terms of the functions z_i of the variables $e^2_{\theta^{\nu\tau}_{\omega\varrho}}$ μ_η^2 m_ζ^2 can be written

$$(1.41) \qquad \frac{(\partial/(\partial k^2/\theta^2)) z_4(k^2/\theta^2, \mu_\eta^2/\theta^2, m_\zeta^2/\theta^2, \eta^2/\theta^2, \langle E\rangle, e^2_{\theta^{\nu\tau}_{\omega\varrho}})}{\theta^2/\mu_\eta^2 - (\theta^2/k^2) z_4(k^2/\theta^2, \mu_\eta^2/\theta^2, m_\zeta^2/\theta^2, \eta^2/\theta^2, \langle E\rangle, e^2_{\theta^{\nu\tau}_{\omega\varrho}})} =$$

$$= \frac{\partial}{\partial \log(k^2/\theta^2)} \log z_3\left(\frac{k^2}{\theta^2}, \frac{\mu_\eta^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, \frac{\eta^2}{\theta^2}, \langle E\rangle, e^2_{\theta^{\nu\tau}_{\omega\varrho}}\right).$$

Being an identity in the variables, it is a functional equation relating z_3 and z_4.

1·5. *Definition of the « group » $G(Z_3 Z_2 Z_2' Z_1^{-1} Z_1'^{-1}, \mu, m)$ and of its restrictions.* – We call $G(Z_3 Z_2 Z_2' Z_1^{-1} Z_1'^{-1}, \mu, m)$ the « group » G in which the functions $\hat{d}_i(k^2/m^2, \lambda^2/m^2, \mu^2/m^2, e^2)$ undergo the change of variables $C^{m(\mu m m)}_{\theta^{\nu\tau}_{\omega\varrho}\eta\zeta}$.

If we were able to solve the functional equation (1.37) we would obtain the Z_i which depends on 11 parameters for $i \geqslant 2$ and 15 for $i = 1$. Of course it is easier, and we would get more physical information, to treat those « groups » for which the number of parameters is reduced. We shall, therefore, introduce restrictions on the general « group » $G(Z_3 Z_2 Z_2' Z_1^{-1} Z_1'^{-1}, \mu, m)$ by fixing the values of some of the parameters entering in the definition of the change of variables C. Of course the demonstration we derive in the preceding Subsection for the general « group » still applies here.

All the possible restrictions to the « group » are the following ones, any combination of these restrictions being also permitted:

One does not change the photon « mass » (fixing η). In G, one drops μ.

One does not change the electron « mass » (fixing ζ). In G, one drops m.

Examples:

$$\eta^2 = \mu^2 \Leftrightarrow \mu_\eta^2 = \mu^2, \qquad \text{or} \qquad \eta^2 = -\infty \Leftrightarrow \mu_\eta^2 = \mu_0^2;$$

$$\zeta^2 = m^2 \Leftrightarrow m_\zeta^2 = m^2, \qquad \text{or} \qquad \zeta^2 = -\infty \Leftrightarrow m_\zeta^2 = m_0^2.$$

When η^2 and ζ^2 are fixed, μ_η and m_ζ are no longer considered as variables, one writes $G_{\mu_\eta m_\zeta}(\ )$.

$\theta^2 = \mu^2$: the « charge » does not depend on $d_3(\theta^2/m^2, \mu^2/m^2, e^2)$; in $G(\ )$ one puts $Z_3 = 1$.

$\nu^2 = m^2$: the « charge » does not depend on $d_2(\nu^2/m^2, \mu^2/m^2, e^2)$; in $G(\ )$ one puts $Z_2 = 1$ (*).

$\omega^2 = m^2$: the « charge » does not depend on $d_2(\omega^2/m^2, \mu^2/m^2, e^2)$; in $G(\ )$ one puts $Z_2' = 1$ (*).

$\tau_{\mathrm{I}}^2 = 0$, $\tau_{\mathrm{II}}^2 = \tau_{\mathrm{III}}^2 = m^2$: the « charge » does not depend on $d_1(\tau^2/m^2, \mu^2/m^2, e^2)$; in $G(\ )$ one puts $Z_1 = 1$ (*).

$\varrho_{\mathrm{I}}^2 = 0$, $\varrho_{\mathrm{II}}^2 = \varrho_{\mathrm{III}}^2 = m^2$: the « charge » does not depend on $d_1(\varrho^2/m^2, \mu^2/m^2, e^2)$; in $G(\ )$ one puts $Z_1' = 1$ (*).

$\nu^2 = \omega^2$: the « charge » depends on $(d_2(\nu^2/m^2, \mu^2/m^2, e^2))^2$; in $G(\ )$ one puts $Z_2 = Z_2'$.

$\tau^2 = \varrho^2$: the « charge » depends on $(d_1(\tau^2/m^2, \mu^2/m^2, e^2))^2$; in $G(\ )$ one puts $Z_1 = Z_1'$.

Also it is possible to build « bound groups » by imposing relations between the parameters of C. For instance if $\zeta^2 = f(\theta^2)$ (a case that we will study in Subsect. 2'2) we write $\widehat{G(Z_3, \ldots, \mu, m)}$.

In practice we are mainly interested in choosing those particular values of the parameters for which the functions we obtain are directly related to the renormalized functions expressed in terms of the bare or of the renormalized constants; for instance by taking $m_\zeta = m$, $\nu^2 = \omega^2$, $\mu_\eta = \mu$, $\tau^2 = \varrho^2 = (0, m^2, m^2)$, the « group » $G_{\mu m}(Z_3 Z_2^2)$ is the one we treated in [6]. The « group » $G_{\mu m}(Z_3)$ is the well-known renormalization group [1].

2. – Integral equations of the « groups » $\widehat{G_{\mu}(Z_3, m)}$ and $G_{\mu_\eta m_\zeta}(Z_3)$.

2'1. *Integral equations of the « groups »* $G_{\mu_\eta m_\zeta}(Z_3)$. – By construction, we can in principle choose for the parameters $-\eta^2$ and $-\zeta^2$ any positive value, even infinite, without introducing divergences in the z_i functions. The case for which such a limit value would introduce divergences can however be handled by using the more complete group $G(Z_3, \mu, m)$ (that we will illustrate in the next Subsection) and letting the variable tend to infinity.

(*) Let us note that when one makes directly the restricted change of variables $(e^2, \mu^2, m^2) \to (e_0^2, \mu_\eta^2, m_\zeta^2)$ which is equivalent to using (1.25), with $\nu^2 = \omega^2 = m^2$ and $\tau^2 = \varrho^2 = (\mu^2, m^2, m^2)$, it is obvious that the z_i do not contain any « old variable » (μ, m) contrarily to the formal and too general expressions (1.28) when $\nu^2 \ldots$ are equal to m^2, μ^2.

The charge of the « groups » $G_{\mu_\eta}(Z_3, m)$, $G_{\mu_\eta m_\zeta}(Z_3)$ is transformed according to the relations

$$(2.1) \qquad e^2_{\theta'} = e^2_\theta / z_3\left(\frac{\theta'^2}{\theta^2}, \frac{m^2_\zeta}{\theta^2}, 1, e^2_\theta\right), \qquad \text{where} \qquad e^2_\theta = e^2/d_2\left(\frac{\theta^2}{m^2}, \frac{\mu^2}{m^2}, e^2\right).$$

a) Nondegenerate case (classical form).

When $m^2_\zeta \neq 0$ or ∞, the z_i-functions of the variables (1.28) can be expressed in terms of ratios

$$\frac{k^2}{\theta^2}, \frac{m^2_\zeta}{\theta^2}, \frac{\lambda^2}{\theta^2}, e^2_\theta, \frac{\mu^2_\eta}{m^2_\zeta}, \frac{\eta^2}{m^2_\zeta}, \frac{\zeta^2}{m^2_\zeta}$$

(one has a symmetrical case when $\mu^2_\eta \neq 0$ or ∞), and are solutions of

$$z_i\left(\frac{k^2}{\theta^2}, \frac{m^2_\zeta}{\theta^2}, \frac{\lambda^2}{\theta^2}, e^2_\theta, \frac{\mu^2_\eta}{m^2_\zeta}, \frac{\eta^2}{m^2_\zeta}, \frac{\zeta^2}{m^2_\zeta}\right) = z_i\left(\frac{\lambda'^2}{\theta^2}, \frac{m^2_\zeta}{\theta^2}, \frac{\lambda^2}{\theta^2}, e^2_\theta, \frac{\mu^2_\eta}{m^2_\zeta}, \frac{\eta^2}{m^2_\zeta}, \frac{\zeta^2}{m^2_\zeta}\right) \cdot$$
$$\cdot z_i\left(\frac{k^2}{\theta'^2}, \frac{m^2_\zeta}{\theta'^2}, \frac{\lambda'^2}{\theta'^2}, e^2_{\theta'}, \frac{\mu^2_\eta}{m^2_\zeta}, \frac{\eta^2}{m^2_\zeta}, \frac{\zeta^2}{m^2_\zeta}\right);$$

it is obvious that the last three variables, consequently, are not involved in that « group » equation and we can drop them, remembering however that the functions z_i depend on these fixed variables. Therefore we have (*)

$$(2.2) \qquad z_i\left(\frac{k^2}{\theta^2}, \frac{m^2_\zeta}{\theta^2}, \frac{\lambda^2}{\theta^2}, e^2_\theta\right) = z_i\left(\frac{\lambda'^2}{\theta^2}, \frac{m^2_\zeta}{\theta^2}, \frac{\lambda^2}{\theta^2}, e^2_\theta\right) \cdot$$
$$\cdot z_i\left(\frac{k^2}{\theta'^2}, \frac{m^2_\zeta}{\theta'^2}, \frac{\lambda'^2}{\theta'^2}, \frac{e^2_\theta}{z_3(\theta'^2/\theta^2, m^2_\zeta/\theta^2, 1, e^2_\theta)}\right).$$

Equation (2.2) is decomposed into

$$(2.3) \qquad z_i\left(\frac{k^2}{\theta^2}, \frac{m^2_\zeta}{\theta^2}, \frac{\lambda^2}{\theta^2}, e^2_\theta\right) = \frac{z_i(k^2/\theta^2, m^2_\zeta/\theta^2, 1, e^2_\theta)}{z_i(\lambda^2/\theta^2, m^2_\zeta/\theta^2, 1, e^2_\zeta)},$$

$$(2.4) \qquad z_i\left(\frac{k^2}{\theta^2}, \frac{m^2_\zeta}{\theta^2}, 1, e^2_\theta\right) = z_i\left(\frac{\theta'^2}{\theta^2}, \frac{m^2_\zeta}{\theta^2}, 1, e^2_\theta\right) \cdot z_i\left(\frac{k^2}{\theta'^2}, \frac{m^2_\zeta}{\theta'^2}, 1, \frac{e^2_\theta}{z_3(\theta'^2/\theta^2, m^2_\zeta/\theta^2, 1, e^2_\theta)}\right)$$

and reciprocally one easily shows that (2.3) + (2.4) $\leftrightarrow$ (2.2) but (2.4) contains fewer variables than (2.2).

(*) $z_1(0, k^2/\theta^2, k^2/\theta^2, m^2_\zeta/\theta^2, 0, \lambda^2/\theta^2, \lambda^2/\theta^2, e^2_\theta)$, $z_1(k^2/\theta^2, 0, k^2/\theta^2, m^2_\zeta/\theta^2, \lambda^2/\theta^2, 0, \lambda^2/\theta^2, e^2_\theta)$, $z_1(k^2/\theta^2, k^2/\theta^2, 0, m^2_\zeta/\theta^2, \lambda^2/\theta^2, \lambda^2/\theta^2, 0, e^2_\theta)$ satisfy the same equation as $z_2(k^2/\theta^2, m^2_\zeta/\theta^2, \lambda^2/\theta^2, e^2_\theta)$; this is consistent with the Ward identity $z_1(0, k^2/\theta^2, k^2/\theta^2, m^2_\zeta/\theta^2, 0, \lambda^2/\theta^2, \lambda^2/\theta^2, e^2_\theta) = z_2(k^2/\theta^2, m^2_\zeta/\theta^2, \lambda^2/\theta^2, e^2_\theta)$.

By differentiating with respect to the first variables the eq. (2.4), one deduces Lie differential equations, which integrated give the integral equations of the « group ».

For $i = 2, \ldots, 5$ one has

$$\frac{\partial}{\partial k^2/\theta^2} z_i\left(\frac{k^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, 1, e_\theta^2\right) = \frac{\theta^2}{\theta'^2} z_i\left(\frac{\theta'^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, 1, e_\theta^2\right) \frac{\partial}{\partial(k^2/\theta'^2)} \cdot$$
$$\cdot z_i\left(\frac{k^2}{\theta'^2}, \frac{m_\zeta^2}{\theta'^2}, 1, \frac{e_\theta^2}{z_3(\theta'^2/\theta^2, m_\zeta^2/\theta^2, 1, e_\theta^2)}\right);$$

defining $\left((\partial/\partial x) z_i(x, y, 1, z)\right)_{x=1} \equiv \varphi_i(\mu, z)$ and taking $k^2 = \theta'^2$ in that equation, one has

$$\frac{\partial}{\partial(\theta'^2/\theta^2)} \log z_i\left(\frac{\theta'^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, 1, e_\theta^2\right) = \frac{\theta^2}{\theta'^2} \varphi_i\left(\frac{m_\zeta^2}{\theta'^2}, \frac{e_\theta^2}{z_3(\theta'^2/\theta^2, m_\zeta^2/\theta^2, 1, e_\theta^2)}\right)$$

and therefore

$$z_i\left(\frac{k^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, 1, e_\theta^2\right) = \exp\left[\int_1^{k^2/\theta^2} \frac{\mathrm{d}t}{t} \varphi_i\left(\frac{m_\zeta^2}{t\theta^2}, \frac{e_\theta^2}{z_3(t, m_\zeta^2/\theta^2, 1, e_\theta^2)}\right)\right], \tag{2.5}$$

and, using (2.3),

$$z_i\left(\frac{k^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, \frac{\lambda^2}{\theta^2}, e_\theta^2\right) = \exp\left[\int_{\lambda^2/\theta^2}^{k^2/\theta^2} \frac{\mathrm{d}t}{t} \varphi_i\left(\frac{m_\zeta^2}{t\theta^2}, \frac{e_\theta^2}{z_3(t, m_\zeta^2/\theta^2, 1, e_\theta^2)}\right)\right]. \tag{2.6}$$

Equation (2.5) can also be written

$$z_3\left(\frac{k^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, 1, e_\theta^2\right) = 1 + \int_1^{k^2/\theta^2} \frac{\mathrm{d}t}{t} z_i\left(t, \frac{m_\zeta^2}{\theta^2}, 1, e_\theta^2\right) \varphi_i\left(\frac{m_\zeta^2}{t\theta^2}, \frac{e_\theta^2}{z_3(t, m_\zeta^2/\theta^2, 1, e_\theta^2)}\right), \tag{2.7}$$

and in particular if $\varphi_3(y, z) = z\psi_3(y, z)$

$$z_3\left(\frac{k^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, 1, e_\theta^2\right) = 1 + e_\theta^2 \int_1^{k^2/\theta^2} \frac{\mathrm{d}t}{t} \psi_3\left(\frac{m_\zeta^2}{t\theta^2}, \frac{e_\theta^2}{z_3(t, m_\zeta^2/\theta^2, 1, e_\theta^2)}\right). \tag{2.8}$$

$\psi_3(m_\zeta^2/t\theta^2, e_\theta^2/z_3(t, m_\zeta^2/\theta^2, 1, e_\theta^2))$ of course depends on the dropped variables μ_η^2/m_ζ^2, η^2/m_ζ^2, ζ^2/m_ζ^2.

If one takes the limits $\theta^2 \to \mu^2 = 0$ and eventually $\zeta^2 = m^2$ one may write

$$d_3\left(\frac{k^2}{m^2}, e^2\right) = 1 + e^2 \int_0^{k^2/m^2} \frac{\mathrm{d}u}{u} \psi_3\left(\frac{1}{u}, \frac{e^2}{d_3(u, e^2)}\right). \tag{2.9}$$

For $i=1$ one deduces by a similar argument

$$\log z_1\left(\frac{q^2}{\theta^2}, \frac{p^2}{\theta^2}, \frac{(p-q)^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, \frac{\varphi^2}{\theta^2}, \frac{\chi^2}{\theta^2}, \frac{\xi^2}{\theta^2}, e_\theta^2\right) = \tag{2.10}$$

$$= \log \frac{z_1(q^2/\theta^2, p^2/\theta^2, (p-q)^2/\theta^2, m_\zeta^2/\theta^2, 1, 1, 1, e_\theta^2)}{z_1(\varphi^2/\theta^2, \chi^2/\theta^2, \xi^2/\theta^2, m_\zeta^2/\theta^2, 1, 1, 1, e_\theta^2)} =$$

$$= \int_{\varphi^2/\theta^2}^{q^2/\theta^2} \frac{\mathrm{d}t}{t}\left[\frac{\partial}{\partial\chi}\log z_1\left(\chi, \frac{p^2}{t\theta^2}, \frac{(p-q)^2}{t\theta^2}, \frac{m_\zeta^2}{t\theta^2}, 1, 1, 1, \frac{e_\theta^2}{z_3(t, m_\zeta^2/\theta^2, 1, e_\theta^2)}\right)\right]_{\chi=1} +$$

$$+ \int_{\chi^2/\theta^2}^{p^2/\theta^2} \frac{\mathrm{d}t}{t}\left[\frac{\partial}{\partial\chi}\log z_1\left(\frac{\varphi^2}{t\theta^2}, \chi, \frac{(p-q)^2}{t\theta^2}, \frac{m_\zeta^2}{t\theta^2}, 1, 1, 1, \frac{e_\theta^2}{z_3(t, m_\zeta^2/\theta^2, 1, e_\theta^2)}\right)\right]_{\chi=1} +$$

$$+ \int_{\xi^2/\theta^2}^{(p-q)^2/\theta^2} \frac{\mathrm{d}t}{t}\left[\frac{\partial}{\partial\chi}\log z_1\left(\frac{\varphi^2}{t\theta^2}, \frac{\chi^2}{t\theta^2}, \chi, \frac{m_\zeta^2}{t\theta^2}, 1, 1, 1, \frac{e_\theta^2}{z_3(t, m_\zeta^2/\theta^2, 1, e_\theta^2)}\right)\right]_{\chi=1}$$

and also the expression for « masses »

$$\begin{cases} \mu_\eta^2 = \mu^2 \exp\left[\displaystyle\int_{\mu^2/\theta^2}^{\eta^2/\theta^2} \frac{\mathrm{d}t}{t}\varphi_{4\mu m}\left(\frac{m^2}{t\theta^2}, \frac{e_\theta^2}{z_3(t, m^2/\theta^2, 1, e_\theta^2)}\right)\right], \\ m_\zeta^2 = m^2 \exp\left[\displaystyle\int_{m^2/\theta^2}^{\zeta^2/\theta^2} \frac{\mathrm{d}t}{t}\varphi_{5\mu m}\left(\frac{m^2}{t\theta^2}, \frac{e_\theta^2}{z_3(t, m^2/\theta^2, 1, e_\theta^2)}\right)\right]. \end{cases} \tag{2.11}$$

Changing $t \to u = t\theta^2/m^2$ and then taking $\theta^2 \to \mu^2 = 0$ one gets

$$m_\zeta^2 = m^2 \exp\left[\int_1^{\zeta^2/m^2} \frac{\mathrm{d}u}{u}\varphi_{5\mu m}\left(\frac{1}{u}, \frac{e^2}{d_3(u, 0, e^2)}\right)\right]. \tag{2.12}$$

b) Degenerate cases. In the preceding cases we supposed that $m_\zeta^2 \neq 0$ or ∞. If now $m_\zeta^2 = 0$ or ∞ we have to replace the variables m_ζ^2/θ^2 by one of the other variables μ_η^2/θ^2, η^2/θ^2, ζ^2/θ^2, and a singular case happens when all these four variables disappear together, for instance, when

$$\alpha) \qquad \eta^2 = \mu^2 = 0 \qquad (\mu^2_{\eta^2=\mu^2} = \mu^2 = \eta^2 = 0)$$

and when furthermore either

$$\beta) \qquad m_0^2 = 0\,,$$

or

$$\gamma) \qquad m_0^2 = \infty\,, \quad i.e.\ m^2_{\zeta^2=-\infty} = \begin{cases} 0\,, \\ \infty\,, \end{cases}$$

or

$$\delta) \qquad m^2 = 0 \,,$$

or

$$\varepsilon) \qquad m^2 = \infty \,, \quad i.e. \; m^2_{\zeta^2 = m^2} - m^2 = \begin{cases} 0 \,, \\ \infty \,. \end{cases}$$

The two cases α) and β), and α) and γ) mean that, when m^2 is given $\neq 0$,

$$(2.13a) \qquad m_0^2 = m^2 d_5\left(\frac{\infty}{m^2}, 0, e^2\right) = \begin{cases} 0 & \beta) \,, \\ \infty & \gamma) \,. \end{cases}$$

They clearly should correspond to different values $e^2_{m_0^2=0}$ or $e^2_{m_0^2=\infty}$ of e^2.

Discarding the case α) and γ), which is unphysical, the case α) and δ) may mean, if m_0^2 is given $\neq 0$, that

$$(2.13b) \qquad m^2 = m_0^2 d_{50}\left(0 = \frac{m^2}{m_0^2}, 0, e_0^2\right) = 0 \,,$$

which also would imply that e^2 has a given value $e^2_{m^2=0}$. In such a case the electron propagator takes the form $[(\gamma \cdot p) d_2(p^2) - m b(p^2)]^{-1}$ (1.3) where $b(0) = 0$. Of course the cases α) and β), and α) and δ) can also occur simultaneously when $d_5 = 0$, the electron propagator having the form $[(\gamma \cdot p) d_2(p^2)]^{-1}$. To deal with this case one can first suppose $m_0^2 = m^2$ and take the limit $m^2 \to 0$ after introducing the parameter θ^2 which fixes the unit of squared mass.

In all these cases z_3 can be expressed in terms of k^2/θ^2 and e_θ^2 and the integral equation corresponding to (2.8) is

$$(2.14) \qquad z_3\left(\frac{k^2}{\theta^2}, e_\theta^2\right) = 1 + e_\theta^2 \int_1^{k^2/\theta^2} \frac{\mathrm{d}t}{t} \chi_3^{(i)}\left(\frac{e_\theta^2}{z_3(t, e_\theta^2)}\right) ,$$

where the index (i) indicates that the various functions $\chi_3^{(i)}$ may, *a priori*, be different from one another; furthermore the observed charges $e^{2(i)}$ of each of these cases, if they exist, should obviously be different.

c) *Hybrid form.* Another interesting form the function z_3 can also be deduced from (2.4) by the following argument. Let us first remark that

$$e^2_{\theta'} = e^2_\theta / z_3\left(\frac{\theta'^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, 1, e_\theta^2\right)$$

does not depend on θ^2 and therefore taking the limit $\theta'^2 \to \mu^2 = 0$ this equality becomes

$$(2.15) \qquad e^2 = e^2_\theta / z_3\left(0, \frac{m_\zeta^2}{\theta^2}, 1, e_\theta^2\right) .$$

Note, however, that if $\theta^2 - \mu^2 = 0$, this equality is an identity, whereas when $\theta^2 \to -\infty$,

$$e^2 = e_0^2 / z_3\left(0, \frac{m_\zeta^2}{-\infty}, 1, e_0^2\right)$$

is a relation between e^2 and e_0^2 if $m_\zeta^2/\infty = 0$ or ∞. When θ^2 is different from these two critical values and if we can invert (2.15) we deduce

$$\frac{m_\zeta^2}{\theta^2} = M(\alpha, \alpha_\theta)\,, \tag{2.16}$$

where $\alpha = e^2/4\pi$ and $\alpha_\theta = e_\theta^2/4\pi$.

Of course M depends on the dropped variables μ_η^2/m_ζ^2, η^2/m_ζ^2, ζ^2/m_ζ^2 of z_3. Since d_5 is not necessarily injective M may also be a set of functions defined for the different intervals of α_θ. Replacing m_ζ^2/θ'^2 by $M(\alpha, \alpha_{\theta'})$ in (2.4) for $i = 3$ one gets a hybrid form of functional equation since it contains old and new charge variables, α and α_θ:

$$z_3\left(\frac{k^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, 1, e_\theta^2\right) = z_3\left(\frac{\theta'^2}{\theta^2}, \frac{\eta_\zeta^2}{\theta^2}, 1, e_\theta^2\right) \cdot z_3\left(\frac{k^2}{\theta'^2}, M(\alpha, \alpha_{\theta'}), 1, e_{\theta'}^2\right). \tag{2.17}$$

From it results the integral equation

$$f_3\left(\frac{k^2}{\theta^2}, \alpha, \alpha_\theta\right) \equiv z_3\left(\frac{k^2}{\theta^2}, M(\alpha, \alpha_\theta), 1, e_\theta^2\right) = 1 - \alpha_\theta \int_1^{k^2/\theta^2} \frac{\mathrm{d}t}{t} H\left(\alpha, \frac{\alpha_\theta}{f_3(t, \alpha, \alpha_\theta)}\right), \tag{2.18}$$

where

$$f_3\left(\frac{k^2}{\theta^2}, \alpha, \alpha_\theta\right) = z_3\left(\frac{k^2}{\theta^2}, \frac{m_\zeta^2}{\theta^2}, 1, e_\theta^2, \frac{\mu_\eta^2}{m_\zeta^2}, \frac{\eta^2}{m_\zeta^2}, \frac{\zeta^2}{m_\zeta^2}\right)$$

does not depend on the parameters ζ and η. Of course, instead of working with our functions d_3 and z_3 defined in Sect. 1, we could have dealt with the function g as it is usually done, and concluded similarly that it satisfies the same type of integral equation involving a function $\bar{H}$ different from H but related to it.

Also we can write (2.18) in the form

$$f_3\left(\frac{k^2}{\theta^2}, \alpha, \alpha_\theta\right) = \exp\left[\int_{k^2/\theta^2}^{1} \frac{\mathrm{d}t}{t} \bar{H}\left(\alpha, \frac{\alpha_\theta}{f_3(t, \alpha, \alpha_\theta)}\right)\right] \tag{2.19}$$

with $\bar{H}(x, y) = yH(x, y)$. From (2.19), one deduces (2.18) and inversely. But in (2.19) by taking $k^2 - \mu^2 = 0$ or $k^2 \to -\infty$, we obtain thereby the ratio e_θ^2/e^2 or e_θ^2/e_0^2. Dividing (2.19) by these quantities one finds the expression of (2.19)

in the limiting cases $\theta^2 \to \mu^2 = 0$ or $-\infty$ which demonstrates that (2.18) is true $\forall\, \theta^2$, provided that (2.15) can be inverted for one value of θ^2.

Let us notice that the expressions (2.14) are particular cases of (2.18) which correspond to those values of α which are solutions of $m_0^2(\alpha) = 0$ or ∞ or $m^2(\alpha) = 0$.

In the case $\theta^2 \to 0$ when $m \neq 0$, (2.18) is an expression for the renormalized function:

$$\left| \begin{array}{l} d_3\left(\dfrac{k^2}{m^2}, \alpha\right) = 1 - \alpha \displaystyle\int_0^{k^2} \frac{\mathrm{d}\tau^2}{\tau^2} H(\alpha, \alpha_\tau)\,, \\ \alpha_\tau = \alpha / d_3\left(\dfrac{\tau^2}{m^2}, \alpha\right). \end{array} \right. \tag{2.20}$$

2·2. *The « bound group »* $\widehat{G(Z_3, m)}$. – After choosing $\eta^2 = +\mu^2$ in order that the propagators are expressed in terms of the observed (and null) mass of the photon, we saw, in the preceding Subsection, how, by fixing $\zeta^2 = +m^2$ or $-\infty$, the propagators are expressed respectively in terms of the observed or of the bare electron mass; depending also on what value of $\theta^2 = 0$ or $-\infty$ we choose, they are also expressed in terms of the observed or bare charge. We would like now to determine the structure of the propagators when they are expressed in terms of electron « mass » and « charge » which may transform *simultaneously* and continuously from the bare to the renormalized ones; we denote by $\widehat{G(Z_3, m)}$ such a « bound group ».

Mathematically one realizes this physical condition by supposing that m_ζ^2 and e_θ^2 both depend on the same parameter, so that $\zeta^2 = f(\theta^2)$.

We suppose f to be bijective, $m^2 = f(\mu^2)$ and $-\infty = f(-\infty)$. The z_i then satisfy

$$\begin{aligned} z_i\left(\frac{k^2}{\theta^2}, \frac{m^2_{f(\theta^2)}}{\theta^2}, \frac{\lambda^2}{\theta^2}, e_\theta^2\right) &= z_i\left(\frac{\lambda'^2}{\theta^2}, \frac{m^2_{f(\theta^2)}}{\theta^2}, \frac{\lambda^2}{\theta^2}, e_\theta^2\right) \cdot \\ \cdot z_i\left(\frac{k^2}{\theta'^2}, \frac{m^2_{f(\theta^2)}}{\theta'^2} z_5\left(\frac{f(\theta'^2)}{\theta^2}, \frac{m^2_{f(\theta^2)}}{\theta^2}, \frac{f(\theta^2)}{\theta^2} e_\theta^2\right), \frac{\lambda'^2}{\theta'^2}, \frac{e_\theta^2}{z_3\left(\theta'^2/\theta^2, m^2_{f(\theta^2)}/\theta^2, 1, e_\theta^2\right)}\right). \end{aligned} \tag{2.21}$$

For $i = 5$ the function z_5 of the variables $f(k^2)/\theta^2$, $m^2_{f(\theta^2)}/\theta^2$, $f(\lambda^2)/\theta^2$, e_θ^2 can be written by re-organizing the variables as $z_5(k^2/\theta^2, m^2_{f(\theta^2)}/\theta^2, \lambda^2/\theta^2, e_\theta^2)$. According to (1.38) it is such that

$$z_5\left(\frac{\lambda^2}{\theta^2}, \frac{m^2_{f(\theta^2)}}{\theta^2}, \frac{\lambda^2}{\theta^2}, e_\theta^2\right) = 1\,.$$

As for eqs. (1.37), a simplification of eq. (2.11) is obtained by defining the

$f_i = z_i|_{\lambda^2=\theta^2}$ which satisfy

$$f_i\left(\frac{k^2}{\theta^2}, \frac{m^2_{f(\theta^2)}}{\theta^2}, e^2_\theta\right) = f_i\left(\frac{\theta'^2}{\theta'}, \frac{m^2_{f(\theta^2)}}{\theta^2}, e^2_\theta\right) \cdot f_i\left(\frac{k^2}{\theta'^2}, \frac{m^2_{f(\theta^2)}}{\theta'^2}, f_5\left(\frac{\theta'^2}{\theta^2}, \frac{m^2_{f(\theta^2)}}{\theta^2}, e^2_\theta\right), \frac{e^2_\theta}{f_3(\theta'^2/\theta^2, m^2_{f(\theta^2)}/\theta^2, e^2_\theta)}\right). \tag{2.22}$$

Taking the derivative of (2.12) with respect to k^2/θ^2, then taking $k^2 = \theta'^2$, and integrating with respect to θ'^2/θ^2 one gets

$$f_i\left(\frac{k^2}{\theta^2}, \frac{m^2_{f(\theta^2)}}{\theta^2}, e^2_\theta\right) = \exp\left[\int_1^{k^2/\theta^2} \frac{\mathrm{d}t}{t} H_i\left(\frac{m^2_{f(\theta^2)}}{\theta^2} f_5\left(t, \frac{m^2_{f(\theta^2)}}{\theta^2}, e^2_\theta\right), \frac{e^2_\theta}{f_3(t, m^2_{f(\theta^2)}/\theta^2, e^2_\theta)}\right)\right], \tag{2.23}$$

or by defining $H_i(x, y, z) = z\bar{H}_i(x, y, z)$

$$f_3\left(\frac{k^2}{\theta^2}, \frac{m^2_{f(\theta^2)}}{\theta^2}, e^2_\theta\right) = 1 - e^2_0 \int_1^{k^2/\theta^2} \frac{\mathrm{d}t}{t} \bar{H}_3\left(\frac{m^2_{f(\theta^2)}}{t\theta^2} f_5\left(t, \frac{m^2_{f(\theta^2)}}{\theta^2}, e^2_\theta\right), \frac{e^2_0}{f_3(t, m^2_{f(\theta^2)}/\theta^2, e^2_\theta)}\right), \tag{2.24}$$

$$m^2_{f(\theta'^2)} = m^2_{f(\theta^2)} \exp\left[\int_1^{\theta'^2/\theta^2} \frac{\mathrm{d}t}{t} H_5\left(\frac{m^2_{f(\theta^2)}}{t\theta^2} f_5\left(t, \frac{m^2_{f(\theta^2)}}{\theta^2}, e^2_\theta\right), \frac{e^2_0}{f_3(t, m^2_{f(\theta^2)}/\theta^2, e^2_\theta)}\right)\right]. \tag{2.25}$$

Of course, one can make in the bound « group » the same argument as in $G(Z_3)$; it leads to eq. (2.18).

2·3. *Notations.* – In Sect. 2·1 we obtained from the « group » $G_{\mu m_\zeta}(Z_3)$ that the function $z_{3\mu m_\zeta}$ expressed in terms of k^2, $m^2_\zeta = m^2$ or m^2_0 and μ^2 satisfies the integral equation (2.8):

$$z_{3\mu m_\zeta}\left(\frac{k^2}{\theta^2}, \frac{m^2_\zeta}{\theta^2}, 1, e^2_\theta\right) = 1 + e^2_\theta \int_1^{k^2/\theta^2} \frac{\mathrm{d}t}{t} \psi_{3\mu m_\zeta}\left(\frac{m^2_\zeta}{t\theta^2}, \frac{e^2_\theta}{z_{3\mu m_\zeta}(t, m^2_\zeta/\theta^2, 1, e^2_\theta)}\right), \tag{2.26}$$

where θ^2 is the renormalization point of Z_3 and e^2_θ the corresponding « charge ». Remarking that, according to (1.28) the $z_{i\mu m_\zeta}$ is related (eventually by means of a change of variable if $\zeta \neq m$) to the ratio of d_i-functions, we have

$$z_{3\mu m_\zeta}\left(\frac{k^2}{\theta^2}, \frac{m^2_\zeta}{\theta^2}, 1, e^2_\theta\right) = \frac{d_3(k^2/m^2, \mu^2/m^2, e^2)}{d_3(\theta^2/m^2, \mu^2/m^2, e^2)},$$

with $d_3(\mu^2/m^2, \mu^2/m^2, e^2) = 1$, and therefore $\lim_{\theta^2 \to \mu^2} z_{3\mu m_\zeta}(k^2/\theta^2, m_\zeta^2/\theta^2, 1, e_\theta^2) = d_3(k^2/m^2, \mu^2/m^2, e^2)$. Taking the limit $\theta^2 \to \mu^2$ in (2.26), changing the variable $t \to u = t\mu^2/m_\zeta^2$ and re-ordering the variables in $z_{3\mu m_\zeta}$ we get, putting $e^2/4\pi = \alpha$, the equation

$$(2.27) \qquad z_{3\mu m_\zeta}\left(\frac{k^2}{\mu^2}, \frac{m_\zeta^2}{\mu^2}, 1, e^2\right) = 1 + e^2 \int_{\mu^2/m_\zeta^2}^{k^2/m_\zeta^2} \frac{\mathrm{d}u}{u} \psi_{3\mu m_\zeta}\left(\frac{1}{u}, \frac{e^2}{z_{3\mu m_\zeta}(u\, m_\zeta^2/\mu^2, m_\zeta^2/\mu^2, 1, e_\theta^2)}\right).$$

After taking the limit $\mu^2 \to 0$ and putting $-k^2 = x^2 > 0$, we rewrite (2.27) by defining F and G functions, respectively for $\zeta^2 = m^2$ and $\zeta^2 = -\infty$:

$$(2.28) \qquad z_3(X, \alpha) \quad \left[z_{3\mu m}\left(\frac{-x^2}{\mu^2}, \frac{m^2}{\mu^2}, 1, e^2\right)\right]_{\mu=0} = 1 - \alpha \int_0^X \int_0^{x^2/m^2} \frac{\mathrm{d}u}{u} F\left(\frac{1}{u}, \frac{\alpha}{z_3(u, \alpha)}\right),$$

$$(2.29) \qquad z_{30}(X_0, \alpha) = \left[z_{3\mu m_0}\left(\frac{-x^2}{\mu^2}, \frac{m_0^2}{\mu^2}, 1, e^2\right)\right]_{\mu=0} = 1 - \alpha \int_0^{X_0 = x^2/m_0^2} \frac{\mathrm{d}u}{u} G\left(\frac{1}{u}, \frac{\alpha}{z_{30}(u, \alpha)}\right).$$

Of course one has $z_3(x^2/m^2, \alpha) = z_{30}(x^2/m_0^2, \alpha)$, which implies

$$(2.30) \qquad F\left(\frac{1}{X}, \frac{\alpha}{z_3(X, \alpha)}\right) = G\left(\frac{1}{X_0}, \frac{\alpha}{z_{30}(X_0, \alpha)}\right)$$

if $m_0^2/m^2 \neq 0$ and is finite.

Also we obtained from $G_{\mu m_\zeta}(Z_3)$ the expression (2.7*a*) or (2.12) for the electron self-energy function that we may write

$$(2.31) \qquad \frac{m_x^2}{m^2} \equiv \frac{m^2(X, \alpha)}{m^2} = \left[z_{5\mu m}\left(\frac{-x^2}{\mu^2}, \frac{m^2}{\mu^2}, \frac{m^2}{\mu^2}, e^2\right)\right]_{\mu=0} =$$

$$= \exp\left[\int_{-1}^{X = x^2/m^2} \frac{\mathrm{d}u}{u} J\left(\frac{1}{u}, \frac{\alpha}{z_3(u, \alpha)}\right)\right], \qquad J(x, \alpha) \equiv \varphi_{5\mu m}(x, e^2),$$

$$(2.32) \qquad \frac{m_x^2}{m^2} = \left[z_{5\mu m_0}\left(\frac{-x^2}{\mu^2}, \frac{m_0^2}{\mu^2}, \frac{m^2}{\mu^2}, e^2\right)\right]_{\mu=0} = \exp\left[\int_{m^2/m_0^2}^{x^2/m_0^2} \frac{\mathrm{d}u}{u} J_0\left(\frac{1}{u}, \frac{\alpha}{z_{30}(u, \alpha)}\right)\right].$$

Of course one has

$$z_{5\mu m}\left(\frac{-x^2}{\mu^2}, \frac{m^2}{\mu^2}, \frac{m^2}{\mu^2}, e^2\right) = z_{5\mu m_0}\left(\frac{-x^2}{\mu^2}, \frac{m_0^2}{\mu^2}, \frac{m^2}{\mu^2}, e^2\right),$$

which implies

$$J\left(\frac{1}{X}, \frac{\alpha}{z_3(X, \alpha)}\right) = J_0\left(\frac{1}{X_0}, \frac{\alpha}{z_{30}(X_0, \alpha)}\right). \tag{2.33}$$

Furthermore we deduced from the « bound group » $G_\mu(Z_3, m)$ expressions for z_3 and for the electron self-energy that we may write, from (2.24),

$$z_3(X, \alpha) = 1 - \alpha \int_0^X \frac{x^2/m^2}{} \frac{\mathrm{d}u}{u} K\left(\frac{1}{u}\frac{\overline{m}^2(u, \alpha)}{m^2}, \frac{\alpha}{z_3(u, \alpha)}\right); \tag{2.34}$$

from (2.25)

$$\frac{m^2_{f(-x^2)}}{m^2} \equiv \left[f_5\left(\frac{-x^2}{\theta^2}, \frac{m^2}{\theta^2}, e^2\right)\right]_{\theta^2=0} \equiv \frac{\overline{m}^2(X, \alpha)}{m^2},$$

$$\frac{\overline{m}^2(X, \alpha)}{m^2} = \exp\left[\int_0^X \frac{\mathrm{d}u}{u} I\left(\frac{1}{u}\frac{\overline{m}^2(u, \alpha)}{m^2}, \frac{\alpha}{z_3(u, \alpha)}\right)\right]. \tag{2.35}$$

It may be remarked that

$$\overline{m}^2\left(-\frac{\theta^2}{m^2}, \alpha\right) = m^2\left(-\frac{f(\theta^2)}{m^2}, \alpha\right) = m^2_{f(\theta^2)}.$$

2·4. *Illustration.* – Let us give in Fig. 1, an illustrative representation, restricted to the $G_\mu(Z_3, m)$ group, of the extended « groups » we constructed.

We consider a reference space of co-ordinates $D(k^2)$ (for fixed k^2), M^2 and G^2, the electron « masses » and « charges ».

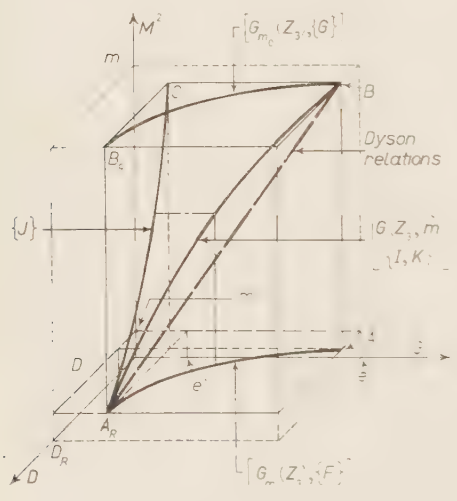

Fig. 1.

The T-transformation characterized by the parameter λ of Subsect. 1·1 corresponds to translations parallel to the D-axes. (For the general « group » there would be five axes and the « group » consists in arbitrary dilatations inside such a space.)

The change of variables $C = (e^2, m^2) \rightarrow (e^2_\theta, m^2_\zeta)$ corresponds, for D fixed, to displacements, determined by the parameters θ and ζ, in a plane parallel to the axes OM^2 and OG^2. (For the general « group » there would be three axes $(OG^2, OM^2, O\mu^2)$ and the displacements are determined by the 15 parameters introduced by the change of variables C

(eq. (1.26)). In our diagram the points A_R and B_0 represent respectively the renormalized and the unrenormalized theories. They are connected by a discontinuous straight line which represents the Dyson's transformation (eq. (1.13)).

When M^2 stays constant and equal to its physical value $M^2 = m^2$, the hyperbola $\overset{\frown}{A_R A_0}$ represents the usual renormalization group, that we denoted by $G_m(Z_3)$, and which, satisfying eq. (2.4), is obtained by taking $\lambda^2 = \theta^2$ in eq. (2.2), which implies the conservation of the invariant $e^2 d_3$.

Associated to the « trajectory » $\overset{\frown}{A_R A_0}$, a function $F(t, \beta)$ was defined by eq. (2.28).

Also, associated with the same group $G_m(Z_3)$ we obtained the formula (2.31) for $m^2(k^2, \alpha)$, which involves the function $J(t, \beta)$.

The existence of such a formula can be either related for k^2 and e^2 fixed, to a T_λ transformation, m^2 being a function of D (formula (2.31)) (and the corresponding « trajectory » is in our diagram the curve $\overset{\frown}{A_R C_0}$ in the plane $G^2 = e^2$), or to a change of variable θ, for D and k^2 fixed, m^2 being then a function of e^{-2}.

In the plane $M^2 = m_0^2$ the hyperbola $\overset{\frown}{B_R B_0}$ represents the « group » $G_{m_0}(Z_3)$; the corresponding function $G(t, \beta)$ being introduced in (2.29).

The « bound group » $G(Z_3, m)$ is represented by a curved trajectory relating A_R to B_0 in the vertical cylinder the base of which is the $\overset{\frown}{A_R A_0}$ trajectory. The corresponding integral eqs. (2.34), (2.35) involve the functions I and K.

The projections of the trajectory $\overset{\frown}{A_R B_0}$ respectively on the planes $G^2 = e^2$, $M^2 = m^2$ and $M^2 = m_0^2$ correspond to the trajectories $\overset{\frown}{A_R C_0}$, $\overset{\frown}{A_R A_0}$ and $\overset{\frown}{B_R B_0}$.

Of course we could also have considered integral equations expressed in term of the bare charge: the corresponding trajectories would correspond to projections of $\overset{\frown}{A_R B_0}$ on the plane $G^2 = e_0^2 \ldots$.

The trajectories $\overset{\frown}{A_R B_R}$ or $\overset{\frown}{A_0 B_0}$ represent simple changes of the mass variable.

Should have we supposed $\lambda^2 = \theta^2$, and treated the equation (2.2), the solution of which is obtained by inserting the solution of (2.4) in (2.3), we would have got, in our Figure, trajectories which cover all of the square of the plane $M^2 = m^2$, and varying ζ one would cover all of the parallelogram the opposite corners of which are A_R and B_0.

3. – The global structures of the functions Z_0 and m_0.

3·1. Z_3.

a) *Unitarity constraint.* We have seen in Sect. 2 (formulae (2.8), (2.14), (2.28), (2.34)) that whatever group we consider the function ζ satisfies an

integral equation of the type

$$z_3\left(\frac{x^2}{\theta^2}, \ldots, \alpha_\theta\right) = 1 - \alpha_\theta \int_{+1}^{x^2/\theta^2} \frac{\mathrm{d}t}{t}\, \zeta\left(\ldots, \frac{\alpha_\theta}{z_3(t, \ldots, \alpha_\theta)}\right),$$

where $x^2 = -k^2$, $\theta^2 > 0$. Before discussing in detail the global properties of the function z_3, we may first restrict the photon propagator to be a physical one, namely such that there is no violation of unitarity.

Writing

$$g^{-1} = 1 - \alpha \Pi(x^2, \alpha) = \left[1 - (x^2 + \mu^2)\alpha \int_0^\infty \frac{\sigma(a)\,\mathrm{d}a}{(a + x^2)}\right]_{\mu^2 = 0} \qquad \text{for } x^2 > 0,$$

one should then have

$$\sigma(a) \geqslant 0 \qquad \text{and} \qquad 1 - \alpha \int_0^\infty \sigma(a)\,\mathrm{d}a \geqslant 0\,.$$

From

$$z_3 \equiv \frac{\partial}{\partial x^2}\, x^2 [1 - \alpha \Pi(x^2, \alpha)]\,,$$

one gets that, for $x^2 > 0$,

$$z_3(x^2, \alpha) = \left[1 - \alpha \int \sigma(a)\,\mathrm{d}a + \alpha \int \frac{a(a - \mu^2)\sigma(a)}{(a + x^2)^2}\,\mathrm{d}a\right]_{\mu^2 = 0} < 1$$

and

$$\frac{\partial z_3(x^2, \alpha)}{\partial x^2} = -\left[\alpha_\theta \frac{1}{x^2}\, \zeta\left(\ldots, \frac{\alpha_\theta}{z_3}\left(\frac{x^2}{\theta^2}, \ldots, \alpha_\theta\right)\right)\right]_{\theta^2 = \mu^2 = 0} =$$

$$2\alpha \left[\int \frac{(a - \mu^2)\, a\sigma(a)}{(a + x^2)^3}\,\mathrm{d}a\right]_{\mu^2 = 0} < 0$$

and therefore

$$\zeta(\ldots \alpha_x) \geqslant 0\,, \qquad \text{for } \alpha \leqslant \alpha_x \leqslant \alpha_0 = \frac{\alpha}{Z_3}\,. \tag{3.1}$$

b) Properties of the hybrid form. Let us now take the form (2.28) which, as we have seen, covers all the possible cases; it may be written

$$\frac{\alpha_\theta}{\alpha_{\theta'}} = 1 - \alpha_\theta \int_{\theta^2}^{\theta'^2} \frac{\mathrm{d}\tau^2}{\tau^2}\, H(\alpha, \alpha_\tau)\,. \tag{3.2}$$

This equation (or the one we could construct for the function g) can, contrary to the form (2.8), be converted into an exact expression which generalizes

that given by GELL-MANN and LOW [1], allowing a simplified discussion,

$$\log \frac{\theta'^2}{\theta^2} = \int_{\alpha_\theta}^{\alpha_{\theta'}} \frac{\mathrm{d}t}{t^2 H(\alpha, t)} . \tag{3.3}$$

Let us first in (3.3) take the limit $\theta'^2 \to 0$ for θ^2 finite; this gives

$$-\infty = \log \left| \frac{0}{\theta^2} \right| = \int_{\alpha_0}^{\alpha} \frac{\mathrm{d}t}{t^2 H(\alpha, t)} . \tag{3.4}$$

The integral diverges then in one of the cases:

$$\text{(3.5)} \quad \begin{cases} 1a) & \alpha = 0 \text{ and } \int_0^x \frac{\mathrm{d}t}{t^2 H(0, t)} \text{ must be divergent, } \forall x \text{ finite, } i.e.\ H(0, 0) \\ & \text{is not necessarily null.} \\ 1b) & H(\alpha, t) \text{ is such that } H(\alpha, \alpha) = 0 . \end{cases}$$

Let us now in (3.3) take the limit $\theta'^2 \to \infty$ for θ^2 finite, one has

$$\infty = \log \left| \frac{\infty}{\theta^2} \right| = \int_{\alpha_0}^{\alpha_{\theta'^2 \to \infty}} \frac{\mathrm{d}t}{t^2 H(\alpha, t)} . \tag{3.6}$$

There are three possibilities for the integral to diverge:

$$\text{(3.7)} \quad \begin{cases} 2a) & \alpha^0 = \alpha_c^0 \text{ if there exists a positive } \alpha_c^0 > \alpha \text{ such that } H(\alpha, \alpha_c^0) = 0. \\ 2b) & \alpha^0 = \infty \ (i.e.\ Z_3 = 0) \text{ and } \int_x^\infty \mathrm{d}t/t^2 H(\alpha, t) \text{ must be divergent } \forall x \\ & \text{finite.} \\ 2c) & \alpha^0 = 0 \text{ which is physically impossible unless simultaneously } \alpha = 0. \end{cases}$$

Cases $2a$) and $2b$) are the well-known possibilities discussed in the literature on the basis of the approximate Gell-Mann–Low formula.

Taking the condition $H(\alpha, \alpha) = 0$ which should be fulfilled $\forall \alpha$, one may consider that, once α is given and providing the universal function $H(\alpha, t)$ is known, the condition $H(\alpha, \alpha_c^0) = 0$ determines $\alpha_c^0(\alpha)$, *i.e.* $Z_3(\alpha)$, the critical value α_c for α being that for which $H(\alpha_c, \alpha_c^0 = \infty) = 0$ and $Z_3(\alpha_c) = 0$. There

is, of course, no contradiction in the existence of two zeros ($\beta = \alpha$ and α_0) of $H(\alpha, \beta)$ since they are at the limit of the domain of positiveness of this function. Some of the critical cases discussed in (2.1*b*) belong to those cases, the coupling constant α being then determined by the constraints on the electron « masses »; they are those for which either

$$m_0 = 0 \quad (m \neq 0) \qquad \text{for some value } \alpha_{m_0=0} ,$$

or

$$m_0 = \infty \quad (m \neq 0) \qquad \text{for some value } \alpha_{m_0=\infty} ,$$

or

$$m = 0 \quad (m_0 \neq 0 \text{ and finite}) \qquad \text{for some value } \alpha_{m=0} .$$

The case $\alpha = 0$ (which may eventually be associated with $\alpha_0 = 0$) corresponds physically to a propagator such that $\alpha_\theta g(x^2, \alpha_\theta)|_{\theta^2 \to 0} \to 0$, *i.e.* $d_3(0, \alpha_\theta) = \infty$.

It likely corresponds to the case $m = m_0 = 0$ ($d_5 = 0$) for which an infra-red-divergent problem occurs at $k^2 = 0$ because of the lack of mass dimension in the theory.

Let us now establish some properties of the functions $H(\alpha, \beta)$ and $z_3(t, \alpha)$.

Supposing $Z_3 \neq 0$ and $H(\alpha, \beta)$ regular around the value $\beta = \alpha_0 < \infty$, *i.e.* such that

$$H(\alpha, \beta) = \big(H(\alpha, \alpha_0) = 0\big) + (\beta - \alpha_0) H'_\beta(\alpha, \alpha_0) + \dots , \qquad H'_\beta \equiv \frac{\partial}{\partial \beta} H(\alpha, \beta) ,$$

and using eq. (3.2) for $\theta^2 = 0$ and $\theta'^2 = x^2/m^2 = t$ one easily obtains

$$z_3(t, \alpha)|_{t \to \infty} = z_3(\infty, \alpha) + \text{const} \left(\frac{1}{t}\right)^{\alpha_0^2 H'_\beta(\alpha, \alpha_0)} ,$$

where, of course, $H'_\beta(\alpha, \alpha_0) < 0$. When z_3 is regular at this limit $\alpha_0^2 H'_\beta(\alpha, \alpha_0) = -1$.

Let us suppose now that $m \neq 0$, that $H(\alpha, \beta)$ is regular around $\beta = \alpha$ and that for $t \to \mu^2/\theta^2$ one has $z_3(t, \alpha) = 1 - \varphi(t)$ with $\varphi(\mu^2/\theta^2) = \mu^2/\theta^2$; one deduces from (2.18) that

$$z_3(t, \alpha) = 1 - \text{const}\ (t)^{\alpha^2 H'_\beta(\alpha, \alpha)} ,$$

where $H'_\beta(\alpha, \alpha)$ should be > 0 to have $\varphi(0) = 0$. If $\mu^2 \neq 0$ the function $z_3(t, \alpha)$ should be regular around $t = 0$ the closest cut occurring for $t = -9\mu^2/m^2$ and therefore $\alpha^2 H'_\beta(\alpha, \alpha) =$ entire number.

Furthermore one may easily verify that due to the unitarity condition $z'_3(t, \alpha)|_{t \to 0} \neq 0$ and therefore $\alpha^2 H'_\beta(\alpha, \alpha) = 1$.

It may also be noticed that if $z_3(t, \alpha)$ and $H(\alpha, \beta)$ are supposed to be regular in $t \to 0$ and $\beta \to \alpha$, it results from eq. (2.18) that $\alpha^2 H'_\beta(\alpha, \alpha) = 1$.

If $\mu^2 = 0$, the $(2N+1)$ photon intermediate states produce cuts in $\Pi(x^2, \alpha)$ and $x^2 = 0$. However they contribute by a series of terms, the most singular of which is of the form $\alpha^3(x^2/m^2)^4 \log x^2/m^2$ and therefore one has again $\alpha^2 H'_\beta(\alpha, \alpha)|_{\mu=0} = 1$.

On the contrary if there exists a value of $\alpha_{m=0}$ such that $m = 0$ $(m_0 \neq 0)$ the $2N$ electron cuts tend to $x^2 = 0$ and there is no more reason for the last relation to be true.

Let us now suppose that we may neglect all the $(2N+1)$ $(N \geqslant 1)$ photon intermediate states in the computation of the actual $\Pi(x^2, \alpha)$ function; then the closest cut comes from the electron-positron pair + N photon intermediate states. It occurs at an energy somewhat below the threshold of these reactions, at the position M_P of lowest triplet positronium bound state (which is stable in this approximation). Near this pole $\propto \Pi(x^2, \alpha)$ takes the form $C\alpha^4 x^2/(x^2 - M_P^2)$ with $M_P = 2M(1 - C(\alpha^2/3))$ and therefore $z_3(x^2, \alpha) \to \infty$ when $x^2 \to M_P^2$.

Using (2.18) one easily establishes that (*)

$$H(\alpha, \beta)\big|_{\beta \to 0} \to (-\text{const})\beta^{-\frac{3}{2}} .$$

Should have we used the function $\tilde{H}$ associated with the conventional g^{-1}-function, which, as is easily seen, possesses the same zeros as $H(\alpha, \beta)$, one would have got

$$\tilde{H}(\alpha, \beta)\big|_{\beta \to 0} \to (-\text{const})\,\beta^{-2} .$$

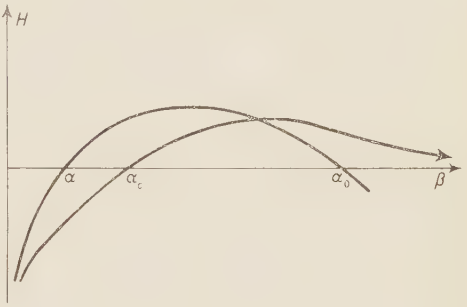

Fig. 2.

Therefore the function $H(\alpha, \beta)$ has qualitatively one of the shapes drawn in Fig. 2, where we admitted that the critical value $\alpha_c(Z = 0)$ corresponds to the maximum value of α.

3˙2. m_0.

a) The comparison of (2.34) and (2.35) with (2.28) and (2.31) implies respectively

$$F\left(\frac{1}{X}, \frac{\alpha}{z_3(X, \alpha)}\right) = K\left(\frac{1}{X}\frac{\overline{m}^2(X, a)}{m^2}, \frac{\alpha}{z_3(X, \alpha)}\right) \tag{3.8}$$

(*) Should we take into account the $(2N+1)$ photon states and use the function $\mathrm{Re}(g)$ which oscillates and is bounded in the domain $0 \leqslant k^2 < 4m^2$ one can see that the branch of the multivalued function $H(\alpha, \beta)$ corresponding to the domain $-\infty < k^2 \leqslant (Mp)^2$ does not reach the axis $\beta = 0$.

and if $f(k^2) \sim k^2$ when $k^2 \to -\infty$

$$J\left(\frac{1}{X}, \frac{\alpha}{z_3(X, \alpha)}\right) = I\left(\frac{1}{X}\frac{\overline{m}^2(X, \alpha)}{m^2}, \frac{\alpha}{z_3(X, \alpha)}\right) \qquad (X \gg 1),$$

therefore

$$\frac{1}{X}\frac{\overline{m}^2(X, \alpha)}{m^2} \quad \varphi\left(\frac{1}{X}, \frac{\alpha}{z_3(X, \alpha)}\right). \tag{3.9}$$

One has also

$$\frac{\partial z_3(X, \alpha)}{\partial x} = -\frac{\alpha}{X} K\left[\frac{1}{X}\frac{\overline{m}^2(X, \alpha)}{m^2}, \frac{\alpha}{z_3(X, \alpha)}\right], \tag{3.10}$$

and putting

$$[\overline{X}(X, \alpha)]^{-1} = \frac{1}{X}\frac{\overline{m}^2(X, \alpha)}{m^2}$$

one gets from (3.8)

$$\frac{1}{X}\frac{\partial \overline{X}}{\partial X} = \frac{1}{X}\left[1 - I\left(\frac{1}{\overline{X}}, \frac{\alpha}{z_3(X, \alpha)}\right)\right].$$

If $z_3(X, \alpha)$ is expressed in terms of $\overline{X}(X, \alpha)$ one has then

$$\frac{\partial z_3(X, \alpha)}{\partial X}\frac{\partial X}{\partial \overline{X}} = -\frac{\alpha}{\overline{X}}\frac{K[1/\overline{X}, \alpha/z_3(X, \alpha)]}{1 - I(1/\overline{X}, \alpha/z_3(X, \alpha))}$$

and therefore

$$z_3(X, \alpha) = 1 - \alpha\int_0^{\overline{X}} \frac{\mathrm{d}v}{v}\frac{K[1/v, \alpha/z_3(X(v), \alpha)]}{1 - I[1/v, \alpha/z_3(X(v), \alpha)]}. \tag{3.11}$$

If we suppose that $\overline{m}^2(X, \alpha)\big|_{X\to\infty}$ reaches quickly enough its asymptotic limit m_0^2, one has then

$$\overline{X}(X, \alpha)\big|_{X\to\infty} = \frac{x^2}{m_0^2} \equiv X_0$$

and therefore

$$z_3(X, \alpha)\big|_{X\to\infty} = 1 - \alpha\int_0^{X_0} \frac{\mathrm{d}v}{v}\frac{K[1/v, \alpha/z_3(X(v), \alpha)]}{1 - I[1/v, \alpha/z_3(X(v), \alpha)]},$$

which, compared to (2.29), gives, for x^2 big enough,

$$G\left(\frac{1}{X_0}, \frac{\alpha}{z_{30}(X_0, \alpha)}\right) = \tag{3.12}$$

$$= \frac{K[1/X_0, \alpha/z_{30}(X_0, \alpha)]}{1 - I[1/X_0, \alpha/z_{30}(X_0, \alpha)]} = \frac{F[1/X, \alpha/z_3(X, \alpha)]}{1 - J[1/X, \alpha/z_3(X, \alpha)]}.$$

The last equality is obtained from (3.8).

Comparing (3.12), (2.30) and (2.33), one deduces that, when x^2 is large enough, if

$$F\left(\frac{1}{X}, \frac{\alpha}{z_3(X, \alpha)}\right) - G\left(\frac{1}{X_0}, \frac{\alpha}{z_{30}(X_0, \alpha)}\right) \cdot K\left(\frac{1}{X_0}, \frac{\alpha}{z_{30}(X_0, \alpha)}\right) \neq 0 ,$$

then

$$J\left(\frac{1}{X}, \frac{\alpha}{z_3(X, \alpha)}\right) = J_0\left(\frac{1}{X_0}, \frac{\alpha}{z_{30}(X_0, \alpha)}\right) = I\left(\frac{1}{X_0}, \frac{\alpha}{z_3(X, \alpha)}\right) \neq 0 . \tag{3.13}$$

It may happen that, in K and I, $m^2(X)$ reaches the limit $m_0^2 = 0$, whereas in G the change of variable $m^2 > m_0^2$ is then meaningless; if G still exists (3.12) is then true.

If it is true that, as in lowest-order perturbation theory, one has

$$G\left(\frac{1}{u}, \beta\right), F\left(\frac{1}{u}, \beta\right), J\left(\frac{1}{u}, \beta\right)\Big|_{u \to \infty} \to G(\beta), F(\beta), J(\beta) , \tag{3.14}$$

one gets

$$F(\beta) = G(\beta)[1 - J(\beta)] , \tag{3.15}$$

a relation which is true in perturbation theory (see (4.3), (4.5), (4.6)):

$$\left(\frac{1}{3\pi} + \frac{\beta}{4\pi^2} + \ldots\right) = \left(\frac{1}{3\pi} - \frac{\beta}{4\pi^2} + \ldots\right)\left(1 + \frac{3\beta}{2\pi} + \ldots\right) .$$

From (3.8) one also deduces the relation

$$\frac{1}{\varphi(\beta)} \frac{\partial \varphi(\beta)}{\partial \beta} = - \frac{1}{\beta^2 G(\beta)} \qquad (\beta \to e_0^2) . \tag{3.16}$$

b) Degenerate cases of (2.1*b*). To each of the degenerate cases of (2.1*b*) correspond constraints on the electron « mass » that we shall make precise.

Case $m^2 = 0$.

(2.11) can be transformed into

$$m_{\zeta'}^2 = m_\zeta^2 \exp\left[\int_{\zeta^2}^{\zeta'^2} \frac{\mathrm{d}\tau^2}{\tau^2} \varphi_{5\mu m}\left(\frac{m^2}{\tau^2}, e_\tau^2\right)\right] , \tag{3.17}$$

where, with the notations of (2.31), $\varphi_{5\mu m} \equiv J$.

Taking in (3.17) the limit, if it exists, $m^2 \to 0$, one finds with the nota-

tion of (3.14) $J(0, y) \equiv J(y)$

$$m^2_{\zeta'} = m^2_\zeta \exp\left[\int_{\zeta^2}^{\zeta'^2} \frac{\mathrm{d}\tau^2}{\tau^2} J(e^2_\tau)\right]. \tag{3.18}$$

Likewise from (2.8) one has

$$\frac{e^2_\theta}{e^2_{\theta'}} = 1 + e^2_\theta \int_{\theta^2}^{\theta'^2} \frac{\mathrm{d}\tau^2}{\tau^2} \psi_{3\mu m}\left(\frac{m^2}{\tau^2}, e^2_\tau\right),$$

where $\psi_{3\mu m} \equiv -F$ in (2.28).

The limit $m^2 \to 0$ gives

$$\frac{1}{e^2_{\theta'}} = \frac{1}{e^2_\theta} - \int_{\theta^2}^{\theta'^2} \frac{\mathrm{d}\tau^2}{\tau^2} F(e^2_\tau) \tag{3.19}$$

with the notation of (3.14).

Then, differentiating (3.18) and (3.19), one gets

$$\frac{\partial}{\partial e^2_{\zeta'}} \log m^2_{\zeta'} = \frac{1}{\zeta'^2} \frac{J(e^2_{\zeta'})}{e^4_{\zeta'}(1/\zeta'^2) F(e^2_{\zeta'})}$$

and therefore

$$m^2_{\zeta'} = m^2_\zeta \exp\left[\int_{e^2_\zeta}^{e^2_{\zeta'}} \frac{J(u)\,\mathrm{d}u}{u^2 F(u)}\right]. \tag{3.20}$$

Equations (3.18) and (3.20), when $\zeta'^2 \to m^2 = 0$, give

$$\int_0^{\zeta^2} \frac{\mathrm{d}\tau^2}{\tau^2} J(e^2_\tau) = \infty \qquad \forall \zeta^2,\ i.e.\ J(e^2) \geqslant 0,$$

$$\int_{e^2}^{e^2_\zeta} \mathrm{d}u\, \frac{J(u)}{u^2 F(u)} = \infty \qquad \forall \zeta,$$

which can be compared to

$$\int_{e^2}^{e^2_\zeta} \mathrm{d}u\, \frac{1}{u^2 F(u)} = \infty,$$

which is given by (3.4) in a different notation.

Of course $J(e^2)$ can take any positive value, including zero, provided $J(e^2_\tau)$ tends like $1/\log \tau^2$ to $J(e^2) = 0$ when $\tau^2 \to 0$.

Case $m_0^2 = 0$.

One can argue about the limit $m_0^2 = 0$ in the same way:

$$m_{\zeta'}^2 = m_\zeta^2 \exp\left[\int_{\zeta^2}^{\zeta'^2} \frac{\mathrm{d}\tau^2}{\tau^2}\, \varphi_{5\mu m_0}\left(\frac{m_0^2}{\tau^2}, e_\tau^2\right)\right], \tag{3.21}$$

taking, if it exists, the limit $m_0^2 \to 0$ in (3.21) one finds, writing

$$\varphi_{5\mu = m_0 = 0}(0, y) \equiv J_0(y)\,,$$

$$m_{\zeta'}^2 = m_\zeta^2 \exp\left[\int_{\zeta^2}^{\zeta'^2} \frac{\mathrm{d}\tau^2}{\tau^2}\, J_0(e_\tau^2)\right], \tag{3.22}$$

where

$$\frac{1}{e_{\theta'}^2} = \frac{1}{e_\theta^2} - \int_{\theta^2}^{\theta'^2} \frac{\mathrm{d}\tau^2}{\tau^2}\, G(e_\tau^2) \tag{3.23}$$

and then

$$m_{\zeta'}^2 = m_\zeta^2 \exp\left[\int_{e_\zeta^2}^{e_{\zeta'}^2} \frac{J_0(u)}{u^2 G(u)}\, \mathrm{d}u\right]; \tag{3.24}$$

in (3.22)

$$\zeta'^2 \to -\infty \int_{\zeta^2}^{-\infty} \frac{\mathrm{d}\tau^2}{\tau^2}\, J_0(e_\tau^2) = -\infty \qquad \forall \zeta,$$

one should have $J_0(e_0^2) \leqslant 0$ and if $J_0(e_0^2) \to 0$ (3.13), then $J_0(e_0^2) \to -0$, but anyway

$$\int^{e_0^2} \frac{J_0(u)}{u^2 G(u)}\, \mathrm{d}u \to -\infty\,.$$

Case $m_0^2 = \infty$.

In (3.21), when $m_0^2 \to \infty$ by a similar argument one gets

$$\int_{\zeta^2}^{-\infty} \frac{\mathrm{d}\tau^2}{\tau^2}\, J_\infty(e_\tau^2) = \infty \qquad \forall \zeta^2,$$

where $\varphi_{5\mu = 0, m_0 = 0}(\infty, y) \equiv J_\infty(y)$, then $J_\infty(e_0^2) \geqslant 0$.

4. – Interpretation and correctness of the equations in perturbation theory.

4·1. *Equation* (2.28). – We noticed in Sect. 1 that the function d_3, which is *not* identical to $(Dx^2)^{-1} \equiv 1 - \alpha \Pi(x^2/m^2, \alpha)$ (where Π is the renormalized photon self-energy function), is however of the same order for $x^2 \gg m^2$; therefore in perturbation theory, the divergent part of $z_3(x, \alpha)$ as $X \to \infty$ is equal to the ultraviolet-divergent part of $1 - \alpha \Pi(X, \alpha)$ and one has for $X \equiv x^2/m^2 \to \infty$

$$\Pi(X, \alpha) \sim \int_A^X \frac{\mathrm{d}u}{u} F\left(\frac{1}{u}, \frac{\alpha}{z_3(u, \alpha)}\right), \qquad \forall A \text{ fixed}; \tag{4.1}$$

$\Pi(x, \alpha)$ is represented by closed-loop graphs in which the internal photon lines are of course clothed by electrons closed loops.

This last effect is taken into account by the presence of the function z_3 (in the form α/z_3 implied by the $G(Z_3)$ group invariance) inside the function F, and perturbation expansion justifies explicitly this interpretation.

One has indeed

$$z_3(X, \alpha) \approx 1 - \alpha \Pi(X, \alpha) \approx$$

$$\approx 1 - \alpha \left[\int_A^X F_0\left(\frac{1}{u}\right) \frac{\mathrm{d}u}{u} + \int_A^X F_1\left(\frac{1}{u}\right) \frac{\alpha}{z_3(u, \alpha)} \frac{\mathrm{d}u}{u} + \int_A^X F_2\left(\frac{1}{u}\right) \frac{\alpha^2}{(z_3(u, \alpha))^2} \frac{\mathrm{d}u}{u} + \dots \right],$$

which, compared with the calculations, up to the order α^3, of the graphs having no clothing of internal photon lines,

$$z_3(X, \alpha) \approx 1 - \left(\frac{\alpha}{3\pi} + \frac{\alpha^2}{4\pi^2} - \frac{\alpha^3}{32\pi^3} + \dots \right) \log X + \text{finite terms}, \tag{4.2}$$

implies

$$\begin{cases} F_0\left(\frac{1}{u}\right)_{u \to \infty} & \frac{1}{3\pi} + o\left(\frac{1}{u}\right), \\ F_1\left(\frac{1}{u}\right)\Big|_{u \to \infty} = & \frac{1}{4\pi^2} + o\left(\frac{1}{u}\right), \\ F_2\left(\frac{1}{u}\right)_{u \to \infty} & -\frac{1}{32\pi^3} + o\left(\frac{1}{u}\right). \end{cases} \tag{4.3}$$

From

$$\alpha^2 \int_A^X F_1\left(\frac{1}{u}\right) \left(1 - \frac{\alpha}{3\pi} \log \frac{u}{A}\right)^{-1} \frac{\mathrm{d}u}{u}$$

we get an α^3 term,

$$-\alpha^3 \log^2\left(\frac{X}{A}\right)\Big/ 4\pi^2 \cdot 3\pi \cdot 2 \ ,$$

which represents correctly ([4b]) the first-order clothing correction of the photon lines inside the α^2 skeleton graphs $\left(2 \ \bigcirc + \bigcirc\right)$.

4·2. *Equations* (2.29) *and* (2.31). – The perturbation expansion of $z_{30}(X_0, \alpha)$, $X_0 \equiv x^2/m_0^2$, was already studied in ([4]) under the hypothesis that $m_0^2 < \infty$; it was there shown that to lowest orders in α

$$z_{30}(X_0, \alpha) \approx 1 - \frac{\alpha}{3\pi} \log X_0 + \frac{\alpha^2}{4\pi^2} \log X_0 + \frac{\alpha^3}{24\pi^3} \log^2 X_0 + \dots \ . \tag{4.4}$$

By comparing the two first orders of this expression with the perturbation expression of (2.29)

$$G\left(\frac{1}{u}, \frac{\alpha}{z_{30}(u, \alpha)}\right) = G_0\left(\frac{1}{u}\right) + G_1\left(\frac{1}{u}\right) \frac{\alpha}{z_{30}(u, \alpha)} + \dots \ ,$$

one gets

$$\begin{cases} G_0\left(\frac{1}{u}\right)_{u\to\infty} = \frac{1}{3\pi} + o\left(\frac{1}{u}\right) , \\ G_1\left(\frac{1}{u}\right)_{u\to\infty} = -\frac{1}{4\pi^2} + o\left(\frac{1}{u}\right) \end{cases} \tag{4.5}$$

(note the change of sign of G_1, with respect to F_1).

The $\alpha^3 \log^2 X_0$ term of z_{30} results, with the correct coefficient, from the first-order clothing of the internal photon line, which is taken into account in the formula

$$-\alpha \int_A^{X_0} \frac{\mathrm{d}u}{u} G_1\left(\frac{1}{u}\right) \alpha \left[1 - \frac{\alpha}{3\pi} \log \frac{u}{A}\right]^{-1} .$$

Also the perturbation expansion of (2.31)

$$\frac{m(X, \alpha)}{m} - 1 = \frac{1}{2} \int_0^X \frac{\mathrm{d}u}{u} \left[J_0\left(\frac{1}{u}\right) + J_1\left(\frac{1}{u}\right) \frac{\alpha}{z_3(u, \alpha)} + \dots\right] + \\ + \frac{1}{8}\left[\int_0^X \frac{\mathrm{d}u}{u} \left[J_0\left(\frac{1}{u}\right) + \dots\right]\right]^2 + \dots$$

gives, up to first order in α,

$$J_0\left(\frac{1}{u}\right) = 0\,, \qquad J_1\left(\frac{1}{u}\right)\Big|_{u\to\infty} = -\frac{3}{2\pi}\,. \tag{4.6}$$

The effect of clothing the photon line with a single electron loop in the α-term of m_0, , is given by the α^2 contribution in

$$\frac{1}{2}\int_1^X \frac{\mathrm{d}u}{u}\left(J_1\left(\frac{1}{u}\right)\frac{\alpha}{z_{30}(u,\alpha)} = -\frac{3\alpha}{2\pi}\left(1+\frac{\alpha}{3\pi}\log u\right)\right) = -\frac{\alpha^2}{2^3\pi^2}\log^2 X\,.$$

This result agrees with the Bialynicka-Birula calculation (7).

This dependence of the self-energy in the photon clothing is similar to the one we deduce for the $G(Z_3Z_2^2)$ « group » (6), which has of course the same z_3 structure as $G_{\mu m_0}(Z_3)$.

To the order α^2 there is also another $\log^2$ contribution coming from

$$\frac{1}{8}\left[\int_1^X J_1\left(\frac{1}{u}\right)\frac{\mathrm{d}u}{u}\right]^2 \to \frac{1}{8}\left(\frac{3}{2\pi}\right)^2\log^2 X$$

and a log contribution coming from (*)

$$\frac{\alpha^2}{2}\int_1^X J_2\left(\frac{1}{u}\right)\frac{\mathrm{d}u}{u}\,.$$

Their sum represents the two graphs and has the correct value given in ref. (7).

4·3. *Perturbation theory to all orders.* – We have just seen that the Taylor expansions of $H(k^2/m^2, z)$ given by formula (4.1) (or of an analogous formula

(7) Z. BIALYNICKA-BIRULA: *Bull. Acad. Pol. Sci. Math.*, **13**, 139 (1965).

(*) Let us mention here an embarrassing question which, related to gauge invariance, is out of the scope of this paper. One knows that m_0/m should be gauge invariant, as is verified in perturbation theory. Therefore to lowest order in α, αJ_1 is gauge invariant. However the term in α^2 generated by αJ_1 in the exponential and which corresponds to an electron self-energy inclusion in the first-order graphs is not gauge invariant, its gauge dependence being cancelled by the $\alpha^2 J_2$ gauge-dependent term.

for the function g) is equivalent to its perturbation graph calculation and we have explicitly checked this agreement in the asymptotic region $k^2 \to -\infty$. Also we saw that the exponential form of m^2 is in agreement with perturbation theory. Of course no conclusion can be reached from lowest-order perturbation on the exact value of m_0 or Z_3, the nullity of m_0 and the infinity of Z_3 being only the reflection of respective choices of the exponential or linear representations (eq. (2.6) and (2.7)).

Indeed we could equally have chosen the exponential representation for Z_3 (eq. (2.5)), whose perturbation expansion may be compared to that of $1/Z_3$ as it can be obtained from its Lehmann's spectral integral; we would then have got that in perturbation theory $Z_3 \equiv 0$!

From the above analysis we saw also that providing that $z_3(t, \alpha)|_{t\to\infty} = Z_3 \neq 0$ one may get from formula (2.29) the asymptotic expression of $\Pi(t, \alpha)$ in terms of α_0; indeed the expansion of Π in terms of α_0 is equivalent to replacing in the bubble graphs the clothed photon propagators by their asymptotic form $1/k^2 Z_3$.

Globally one gets thereby that, providing $m_\zeta^2 \neq 0$,

$$z_3\left(\frac{k^2}{m_\zeta^2}, \alpha\right)_{k^2\to-\infty} \to Z_3 = 1 - \alpha \int_{\mu^2}^{k^2\to-\infty} \frac{\mathrm{d}u}{u}\, \psi_3\left(\frac{m_\zeta^2}{u}, \alpha_0\right),$$

and if one admits the *hypothesis* that

$$\psi_3\left(\frac{m_\zeta^2}{u}, \alpha_0\right)_{u\to\infty} \to \psi_3(0, \alpha_0) \neq 0 \text{ or } \infty,$$

then

$$Z_3 = 1 - a\psi_3(0, \alpha_0)\left[\log \frac{k^2}{\theta^2}\right]_{k^2\to-\infty}.$$

This can only be consistent if $\alpha_0 = \alpha_{0c}$ is such that $\psi_3(0, \alpha_{0c}) = 0$, the bare charge being then independent of α. If we develop $\psi_3(0, \alpha_0)$ in perturbation then $\Pi(k^2/m_\zeta^2, \alpha_0)$ is, in each order in α_0, no more divergent than a simple log.

However the above hypothesis has never been proved in general; it has only been explicitly checked up to order α_0^2.

Let us now take the singular cases of (2.1*b*) for which such a hypothesis is useless since the

$$X_3^{(i)} \qquad X_3^{(i)}\left(\frac{\alpha^{(i)}}{z_3(t)}\right).$$

By similar argument one gets then that one should have $X_3^{(i)}(\alpha_0^{(i)}) = 0$.

But contrarily to the preceding case, that conclusion means that for each value $\alpha = \alpha^{(i)}$ for which the theory is singular (m^2 or $m_0^2 = 0$ or ∞) then the corresponding $\alpha_0^{(i)}$ are determined by $X_3^{(i)}(\alpha_0^{(i)}) = 0$. That conclusion is in agreement with the analysis of Subsect. 3·2, *b*).

Furthermore if we develop $X_3^{(i)}(\alpha_0)$ in perturbation theory in α_0 one gets for the asymptotic form of $\Pi(k^2)$, a series $\sum a_0^n c_n \log k^2$, a result that JOHNSON *et al.* proved to all orders, by direct graph estimates in the case $m_0 = 0$ (*i.e.* $S'(p)|_{p\to\infty} \to A/i\gamma p$), the relation $m_0 = 0$ being considered in their paper as an identity in α.

However the function $\psi_3^{(i)}(\alpha_0^{(i)})$ being then identical to the function $H(\alpha^{(i)}, \alpha_0^{(i)})$ that we described in Subsect. 3·2 *b*) we see at once that a perturbation expansion in β of $H(\alpha^{(i)}, \beta)$ is very likely meaningless due the singularity of H for $\beta \to 0$.

5. – Global versus perturbation properties of Z_3.

Let us now summarize briefly the situation which comes about by comparing the global results of Sect. 3 with those of Sect. 4. We obtain in Sect. 3 that, because of the properties of the universal function $H(\alpha, \beta)$ which suffices to characterize every possible electrodynamics (except perhaps that for which $m^2 = m_0^2 = 0$), the bare charge α_0 *is in principle that function of* α such that $H(\alpha, \alpha_0) = 0$. This conclusion is also confirmed by the different degenerate cases of (2.1*b*) which, if they exist, should correspond to different and particular values $\alpha^{(i)}$ of α, to which correspond different functions $X_3^{(i)}$ and in principle also different values of α_0. However we have seen that opposite conclusions can be reached in the general case if one uses the representation of Z_3 by means of the usual function $F(1/t, \beta \equiv \alpha/z_3)$ to compute α_0 when α is given. Indeed, if the limit

$$F\left(\frac{1}{t}, \beta\right) \xrightarrow[t\to\infty,\ \beta=\text{const}]{} F(0, \beta)$$

exists and is different from zero the critical value $\beta = \alpha_0$ is the solution, if it exists, of $F(0, \alpha_0) = 0$ and is *independent of* α, a conclusion often stated as a main consequence of the renormalization group.

Though perturbation theory exhibits some arguments in that direction, we have seen that they are far from constituting a proof. This conclusion being not only paradoxical as it invalidates the underlying philosophy of renormalization, and frustrating as it is then difficult to imagine what may determine the observable constants, let us use the representation of z_3 by which it is easy to get a different conclusion, and see, by means of an example, where might be the failure from which results that paradox. Since one can obtain the

function $H(\alpha, \alpha_\tau)$ by substituting (2.16)

$$\frac{m^2}{\tau^2} = \frac{1}{t} M(\alpha, \alpha_\tau)$$

into F, it is clear that depending on whether eq. (2.16) implies at the limit $\tau^2 \to -\infty$ that $\alpha_0 = \text{const}$ or $\alpha_0 = \alpha_0(\alpha)$, one will get correspondingly specific examples in agreement or not with the classical paradoxical result.

Let us therefore take the reasonable asymptotic form

$$\frac{1}{\alpha_\tau} \equiv \frac{1}{\alpha} z_3\left(\frac{\tau^2}{m^2}, \alpha\right) = K(\alpha) - A(\alpha)/t^{B(\alpha)},$$

then (2.16) is

$$\frac{1}{t} = \left[\left(-\frac{1}{\alpha_\tau} + K(\alpha)\right) \Big/ A(\alpha)\right]^{1/B(\alpha)},$$

and we see that both cases are found depending on the choice

i) $K(\alpha) = \text{const} \to \alpha_0 = \text{const}$,

ii) $K(\alpha) = Z_3(\alpha)/\alpha \to \alpha_0(\alpha) = K(\alpha)^{-1}$.

In order to simplify the calculation let us suppose more specifically that

$$A(\alpha) \equiv \frac{A}{\alpha}, \qquad B = 1, \qquad K(\alpha) = \frac{c - \lambda\alpha}{\alpha},$$

one then easily computes that

$$H\left(\alpha, \frac{\alpha}{z}\right) \equiv \left(\frac{z}{\alpha} - \frac{c - \lambda\alpha}{\alpha}\right) \qquad \text{and} \qquad F\left(\frac{1}{t}, \frac{a}{z}\right) \equiv \frac{A}{A + Ct}\left(\frac{z}{\alpha} + \lambda\right)$$

and one sees that if $c = 0$ (case i)), $\alpha_0 = -1/\lambda$ is the zero of $F(0, \alpha_0)$, whereas if $c \neq 0$ (case ii)), $\alpha_0 = 1/c - \lambda\alpha$ is the solution of $H(\alpha, \alpha_0) = 0$, while $F(0, \alpha_0) \equiv 0$, $\forall \alpha_0$; furthermore, if we ignored that, when $c \neq 0$, $A/(A + Ct)|_{t\to\infty}$ were null, the zero of the function $F(0, \beta)$ is not related to the true value of α_0.

Coming now back to the actual electrodynamics for which we may temporarily take for granted that in each order of perturbation theory in α_0 of $\Pi(k^2)$ there is at most a $[\log k^2]^1$ divergent term, we see that the paradox could be solved if there exist functions such that

$$F\left(\frac{1}{t}, \beta\right)_{t\to\infty} \to 0 \qquad \forall\beta$$

and whose Taylor expansion in β is such that

$$F\left(\frac{1}{t}, \beta\right) = \sum_n C_n(t)\beta^n$$

with

$$C_n(t)|_{t\to\infty} = C_n + o\left(\frac{1}{t}\right)$$

for a set of values of n.

There are, in fact, many examples of functions having those properties. Consider for instance the class of functions of the form

$$F\left(\frac{1}{t}, \beta\right) = \frac{A(\beta) + B(\beta)\, t^{-C(\beta)/C(0)}}{1 + D(\beta)\, t^{-E(\beta)/E(0)}},$$

where the functions A ,..., E admit convergent perturbation expansions for $\beta < \beta_c \neq 0$.

$PT(F)$, the perturbation theory expansion in β of F, is then, with

$$PT\left(\frac{C(\beta)}{C(0)}\right) = 1 + c_1\beta + c_2\beta^2 + \ldots \qquad \text{and} \qquad PT\left(\frac{E(\beta)}{E(0)}\right) = 1 + e_1\beta + e_2\beta^2 + \ldots,$$

$$PT\left(F\left(\frac{1}{t}, \beta\right)\right) = PT(A(\beta)) + PT\left[B(\beta)\frac{1}{t}\exp\left[-(c_1\beta + c_2\beta^2 + \ldots)\log t\right]\right.$$

$$\left. + A(\beta)D(\beta)\frac{1}{t}\exp\left[-(e_1\beta + e_2\beta^2 + \ldots)\log t\right]\right] + o\left(\frac{1}{t^2}\right).$$

Therefore

$$\lim_{t\to\infty} PT\left(F\left(\frac{1}{t}, \beta\right)\right) \to PT(A(\beta)) = A(\beta).$$

However the functions F may have all the desired properties providing the functions A, B, C, D, E belong to the classes that we now define:

1) One may have regularity condition of z_3 near $t \to 0$ if

a) $$C(\alpha)/C(0) > 0, \qquad E(\alpha)/E(0) > 0,$$

b) $$E(\alpha)/E(0) - C(\alpha)/C(0) = 1,$$

c) $$B(\alpha)/D(\alpha) > 0.$$

2) One may have the existence of a relation between α and α_0, *i.e.*

$$F\left(\frac{1}{t}, \beta_t\right)\Big|_{t\to\infty} = 0 \qquad \forall \beta_{t\to\infty} = \alpha_0 \in \mathscr{E}_0 ,$$

if

a) $$C(\alpha_0)/C(0) < 0 , \qquad E(\alpha_0)/E(0) < 0 ,$$

b) $$E(\alpha_0)/E(0) < C(\alpha_0)/C(0) .$$

3) One may have the unitarity condition, *i.e.*

$$F\left(\frac{1}{t}, \beta_t\right) > 0 \qquad \forall t > 0 ,$$

if

$$A(\beta) , \; B(\beta) , \; D(\beta) > 0 \qquad (\alpha < \beta < \alpha_0; \; \alpha \in \mathscr{E}, \; \alpha_0 \in \mathscr{E}_0) .$$

Once such a type of function is given, it is clear that providing one can solve the equation

$$u' \quad -\frac{1}{t} F \quad \frac{1}{t}, u\Big) ,$$

where $u \quad z/\alpha = 1/\beta$, with a boundary condition (say $u(1) = \gamma$) one can compute

$$u(0, \gamma) \equiv \frac{z(0, \alpha)}{\alpha} = \frac{1}{\alpha(\gamma)} ,$$

which should be finite and

$$u(\infty, \gamma) \equiv \frac{z(\infty, \alpha)}{\alpha} = \frac{1}{\alpha_0(\gamma)}$$

and therefore deduce a relation $\alpha_0 = f(\alpha)$.

Of course the conditions we gave are only necessary conditions and in particular, once reasonable functions A ,..., E are given, the domains $\mathscr{E}$ and $\mathscr{E}_0$ would likely be reached only for a limited set of values of the initial-value parameter γ.

This example also illustrates the essential difference which may exist between taking the limit $m \to 0$ or having no mass in the theory ($m \quad m_0 \quad 0$),

which amounts to dropping the $(1/t)$-dependent terms in F, therefore retaining only $A(\beta)$. It also indicates that the computation by perturbation theory of the sole function $A(\beta)$ is irrelevant for the determination of α_0.

From a mathematical point of view one can easily verify that the paradoxical and, in the present example, wrong result of perturbation theory in β is due to using a series expansion of $F(1/t, \beta)$ in a domain in which it does not converge.

6. – Conclusion.

It has always been very mysterious that the usual renormalization group, which is always treated in a very rough way, gives results in agreement with perturbative computational facts (for instance that the α and α^2 terms of $Z_3(\alpha)$ are no more divergent than a logarithm once the $(\alpha)^0$-term is assumed), and furthermore implies a specific global and physical consequence (that α^0 is independent of α) which is in direct contradiction with the renormalization postulate from which the group is supposed to be constructed.

In the Sect. 1 of that paper we « demystified » that « group », by showing that the most general and exact formulation of the renormalization « groups » contains almost nothing (an arbitrary change of scale plus a suitably chosen change of variables on well-defined Green functions).

As a consequence of our extension of the usual group we were able to demonstrate in a trivial way Johnson's result [3], which required a complicated computational analysis, that when the electron bare mass is null the photon vacuum polarization tensor does not contain in perturbation theory in α_0 terms more divergent than a single logarithm.

Also owing to the fact that our exact formulation amounts to trivial operations on given propagators and therefore cannot have any physical consequence, we could see that the paradoxical consequence of the former and approximate formulation rests entirely upon the implicit hypothesis that, in the asymptotic region, the variable m/t can be dropped without modifying the global structure of the kernel of the group integral equations and we were then able to overcome that troublesome conclusion. One should also note that, curiously enough, all the conclusions of the former renormalization group and all our discussions of Sect. 3 rest upon the renormalization group integral equations for $\theta^2 = 0$ (eqs. (2.9) and (2.20)), which can be deduced from simple dimensional considerations. Indeed we can always write for $m^2 \neq 0$

$$d_3\left(\frac{k^2}{m^2}, e^2\right) \equiv \varphi_1\left(\frac{k^2}{m^2}, e^2\right)$$

and

$$\frac{\partial}{\partial k^2} d_3\left(\frac{k^2}{m^2}, e^2\right) \equiv \frac{1}{k^2} \varphi_2\left(\frac{k^2}{m^2}, e^2\right);$$

replacing in φ_2 the variables e^2 (respectively m^2/k^2) expressed from φ_1 in terms of e^2/d_3 and m^2/k^2 (respectively e^2), and using the condition $d_3(0, e^2) = 1$, one gets at once the integral equations (2.9) (respectively (2.20)). If $m = 0$ a similar consideration can be done by using m_0^2/k^2 providing $m_0^2 \neq 0$. These equations are of course *consistent* with the group but they *do not imply it*, since one cannot start from (2.9) to obtain (2.8) without postulating in a self-consistent manner the transformation law of the generalized charge (2.1).

One may then inquire if any use and interest remains in the renormalization groups.

a) First of all it may be that some interesting consequences (for instance approximation methods) can be deduced from the functional or integral equations of $\theta^2 \neq 0$ which finally were never fully used because of the difficulty of their interpretation since in practice one can only recognize the perturbation expansions of propagators in terms of bare or observed parameters.

b) As was illustrated here in the case of the $G(Z_3)$ groups for which $\alpha/z_3(t, \alpha)$ comes out as a natural parameter of development, the renormalization group is a very powerful means to obtain exact information on the properties and on the limits of validity of the various approximations one may be tempted to use when computing propagators in some given perturbation scheme.

c) Finally it may happen that the representations of the propagators by the integral equations of the renormalization groups appear to be useful if the kernel functions turn out to have simpler structures than the propagator.

RIASSUNTO (*)

Da una meticolosa chiarificazione degli assiomi da cui risultano i « gruppi » di rinormalizzazione si costruiscono i « gruppi » più generali che includono la rinormalizzazione delle masse. Si discute in dettaglio il caso particolare in cui si hanno solo il campo dei fotoni e la rinormalizzazione della massa dell'elettrone. Un'equazione integrale esatta, che può essere posta in una forma che generalizza quella data da Gell-Mann e

(*) *Traduzione a cura della Redazione.*

Low, permette di dimostrare che una dipendenza della carica nuda α_0 da quella osservata α è, contrariamente al paradosso spesso affermato, coerente col gruppo di rinormalizzazione. Si dimostra che nei casi in cui la massa nuda o quella osservata dall'elettrone è nulla o infinita, che corrisponderebbe a determinati valori di α, lo sviluppo perturbativo formale in α del tensore di polarizzazione del vuoto del fotone non è più divergente di una singola potenza di un logaritmo. Infine si determina dove stia l'errore nelle argomentazioni che portano al paradosso suddetto e se ne discutono le conseguenze.

Перенормированные «группы», расширенные на массы; Применение к структуре перенормированных констант Z_3 и m_0.

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M. ASTAUD, *et al.*
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